Transformations for the Twenty-First Century

Symmetries, Intervals, and the Principle of Transformational Closure

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Abstract This article addresses three interlocking topics. First, it defines *voice-leading transformations* that model the specific contrapuntal moves available to musicians. These are the intervals corresponding to the symmetries of voice-leading geometry, encompassing voice exchanges, transpositions along the chord, repeating contrapuntal patterns, and Richard Cohn's version of the neo-Riemannian transformations. Second, it defines a class of ad hoc transformations known as *contrapuntal interchanges*, which generalize voice-leading transformations using the retrograde. This allows us to reformulate neo-Riemannian theory without inversion, avoiding dualism while encompassing a wider range of progressions. Contrapuntal interchanges can also be used to form approximate sets such as "triad" or "fourth chord," providing a more flexible alternative to standard set theory. Finally, the article uses these technical issues to explore larger questions about the relation between abstract mathematics and concrete musical practice. It suggests that contemporary theory is still in the grip of a modernist ideology that prioritizes systematicity over musical relevance; this ideology is as characteristic of Riemann as it is of Schoenberg.

Keywords transformations, voice leading, intervals, Riemann, perceptibility, modernism

TWENTY-FIRST-CENTURY musicians might feel there is something strange about traditional pitch-class transformations. How should one apply the inversion $\mathbf{I}_0$ to the arpeggio in Figure 0.1a? An intuitive answer is given by Figure 0.1b, but in principle Figure 0.1c is equally good: both map C to C, E to A♭, and G to F, in accordance with the definition of $\mathbf{I}_0$; they distribute those pitch classes into different octaves, but traditional set theory is completely agnostic about that issue. A similar point could be made about Figures 0.1d–e, which apply the neo-Riemannian "relative transform" to the initial arpeggio. One might think that Figure 0.1d is the natural or appropriate answer, but nothing in neo-Riemannian theory privileges it over the alternative. For all the talk about "parsimonious voice leading," that distinction is relegated to the unformalized domain of intuition and practice.

This may not have been a pressing issue in the days when avant-garde composers used pencil and paper to create pointillistic soundscapes. Contemporary musicians, however, are more likely to be concerned with voice leading, and more

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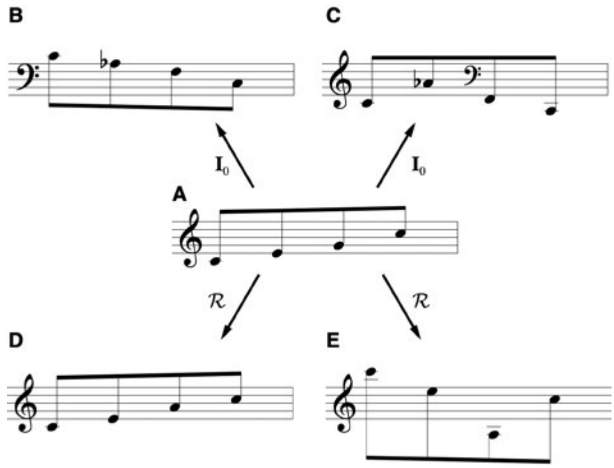

Figure 0.1. In standard theory, the pitch-class inversion I_0 can turn A into either B or C. In neo-Riemannian theory, the R transform can turn A into either D and E.

likely to use notation programs, DAWs, or specialized music software (e.g., Pure Data [Pd], Max/MSP, SuperCollider, music21, Opusmodus, or OpenMusic). In this context transformational ambiguity is a genuine problem: Software needs to behave predictably, and users do not want their computers choosing randomly between Figures 0.1b and c. How can we specify one transformation rather than the other? Theoretically, it is awkward to have just one label for such different-sounding transformations: When doing computational corpus analysis, for example, we might want to distinguish the idiomatic move from Figure 0.1a to Figure 0.1d from the decidedly nonidiomatic move to Figure 0.1e. How should we label these different transformations? Such questions are, by necessity, being answered by programmers around the world: Indeed, in Michael Cuthbert's music21, the command "analysis.neoRiemannian.R" turns Figure 0.1a into Figure 0.1d, and not Figure 0.1e. What operation is the program implementing? Music theory should have an answer.

The first goal of this article is to describe transformations consistent with this nascent common practice—transformations that are both sensitive to voice leading and precise enough to be applied algorithmically. I identify two different kinds of operation. *Voice-leading transformations* are the intervals corresponding to the symmetries of voice-leading geometry, inherent in the very definition of those symmetries. These include voice leadings, voice exchanges, transpositions along the chord, transpositions, repeating contrapuntal patterns, and contextual inversions. *Contrapuntal interchanges* are ad hoc transformations that group together voice-leading transformations related by temporal retrograding; thus, a single contrapuntal interchange takes *A* to *B* and *B* back to *A*, whether those set-classes are related or not. (Cf. Figure 0.2d, which connects unrelated set-classes

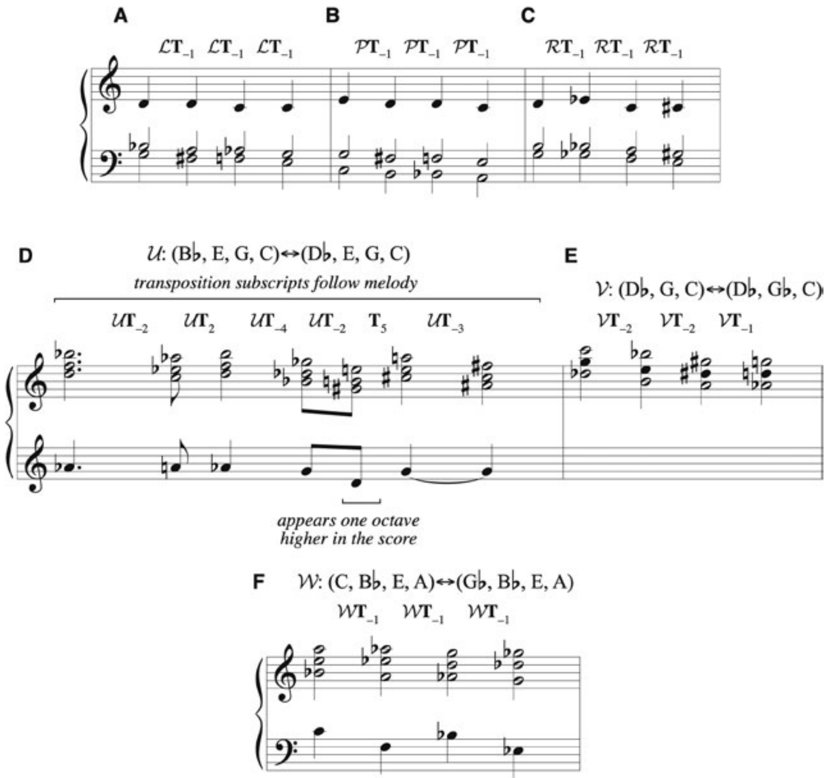

Figure 0.2. Contrapuntal interchanges combined with transposition; open noteheads show parallel voices. A–C. The three basic neo-Riemannian transformations combined with descending-semitone transposition. D. A passage from R84 of *The Rite of Spring* combining a contrapuntal interchange with transpositions determined by the melody. E. The upper voices at R109 of *The Rite of Spring*. F. A progression found in Berg, Debussy, and jazz, combining a contrapuntal interchange with descending semitone transposition (Stuckenschmidt 1965).

sharing a major-triad subset, to Figure 0.2e, which connects inversions; the two passages are musically similar and are found in the same piece.) Contrapuntal interchanges can also be used to create equivalences between set-classes, allowing us to define generic musical objects such as "triads," "minor scales," and "quartal tetrachords" (Section 6); this yields more flexible harmonic categories transcending the binary opposition between absolute identity and absolute difference.

A second goal is to explore what I call *the principle of transformational closure*. That principle states that if *A* is a significant (i.e., comprehensible, musically relevant, analytically fruitful) musical transformation and *B* is a significant musical transformation, then the combination *AB* is also significant. Both voice-leading transformations and contrapuntal interchanges challenge this principle. Figure 0.3 defines two contrapuntal interchanges connecting minor triads and

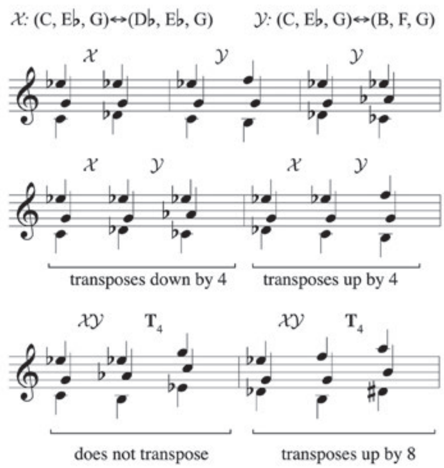

Figure 0.3. The contrapuntal interchange *XY* sends C minor down by major third to A♭ minor and {E♭, G, D♭} up by major third to {G, B, F}. The transformation *XY* $\mathbf{T}_4$ sends C minor to C minor and {E♭, G, D♭} to {B, D♯, A}.

026 trichords; one raises the triad's root by semitone while the other moves root and third by contrary motion to create an incomplete dominant seventh; they apply the inverse voice leadings to the 026 trichord. The combination *XY* transposes the minor triad by four descending semitones and the 026 trichord by four *ascending* semitones. I find it hard to hear *XY* as a coherent musical operation that "does the same thing" to 037 and 026 trichords, and hard to imagine any sensible analysis treating it as such. Even more problematic is *XY*$\mathbf{T}_4$, which transposes one trichord up by eight semitones while not transposing the other at all. (Note that although I have labeled this transformation as the product of *X*, *Y*, and $\mathbf{T}_4$, it is mathematically a single operation, just like 3×4 is the number 12.) Again, I find it exceedingly difficult to understand these operations as "doing the same thing" to their respective sonorities, and difficult to imagine an enlightening analysis interpreting the music in this way.

This is in many ways a familiar dilemma in neo-Riemannian theory, which considers a subset of the transformations discussed in this article. Broadly speaking, neo-Riemannian theorists have explored two resolutions of this tension, one rejecting ordinary transposition (Klumpenhouwer 1994, 2000; Cohn 2012), and the other accepting arbitrary combinations of transposition and contrapuntal interchanges (Hook 2002, 2023). Here I will propose a third and to my knowledge novel solution, which is to embrace *some* but not *all* combinations of transformations; in particular, I will argue that transformations like *X*$\mathbf{T}_2$ (appearing in Figure 4.3 below) are musically significant in a way that transformations like *XY*$\mathbf{T}_4$ are not. This requires a thorough rethinking of the foundations of transformational theory and a more explicit reckoning with its modernist inheritance.[1]

1 This article provides a theoretical underpinning for the examples in sec. 1.1 of Tymoczko 2023c. I arrived at those examples intuitively, without quite realizing that they raised the methodological questions discussed here.

1. Arguments for Transformational Closure

The principle of transformational closure raises two distinct music-theoretical questions. The first involves *composition*: Is it always true that two musically meaningful transformations *A* and *B* combine to produce a musically meaningful composite *AB*? The second involves *group theory*: If *A* and *B* each belong to transformation groups, and if *A* and *B* can be meaningfully combined, when should we construct a larger transformation group containing both *A* and *B*? We will consider these questions in the next two sections.

A sweeping endorsement of transformational closure appears as "Principle 2" in Guerino Mazzola's (2002: 160) *Topos of Music*: "Concatenation of two *musically meaningful* symmetries or morphisms yields a *musically meaningful* result" (emphasis added). This would be important if true, as it would allow us to generate a wealth of meaningful composites from any collection of meaningful generators. Yet transposition by one semitone is musically meaningful while transposition by a billion semitones clearly is not, and a billion-semitone transposition can be obtained by repeatedly concatenating one-semitone transpositions. More directly, Figure 0.3 shows that meaningful transformations can combine to produce composites whose relevance is quite obscure. This refutes the general claim that the concatenation of meaningful operations is *always* meaningful. This is hardly a surprise: music is limited by physics, psychology, and convention, and not beholden to mathematical stipulations such as Mazzola's Principle 2.

An alternative justification focuses on *definability* rather than significance. For if we know how to apply transformation *A* to every object in a set ***S***, and if we know how to apply transformation *B* to every object in that set, and if *A* and *B* have domain and range in ***S***, then we know how to apply the transformation *AB* to every object in that set. That is true as a matter of logic: The transformation *AB* exists in an abstract mathematical sense. But it does not follow that human beings can *perceive* or *understand* it as a unified operation. By way of analogy, moving the queen diagonally and horizontally are both legal chess moves, but moving the queen diagonally and then horizontally is not a legal chess move. *Mathematically* we know how to combine the moves of bishop and rook, but this tells us nothing about whether the combination is legal within the game ("comprehensible as a single chess move"). Definability does not imply comprehensibility, grammaticality, or anything similar.

A third and subtler line of argument comes from David Lewin, who proposed that the concatenation of *conceivable* transformations is itself *conceivable* (Lewin [1987] 2007: 27). This raises more complicated issues, as it is not obvious how conceivability relates to music making. But I think there are at least three reasons to doubt that the concept can bear the weight Lewin places on it.

Reason 1. It is not always clear whether we are successfully conceiving something.

Humans can perceive about ten octaves of sound, from 20Hz to 20,000Hz. Beyond that, soundwaves are bounded by physics: At the low end, by the size of the transmitting medium, which on Earth is the circumference of the planet; at the high end, by the frequency at which Brownian motion produces collisions in the particles of the medium. This gives about fifty octaves of physically existing sound on earth.[2] Further high-energy limits arise from quantum physics and relativity.

So what does it mean to imagine a soundwave whose period is a billion times the age of the universe? What is the transmitting medium? And when we imagine extraordinarily high-frequency sounds, are we considering the thermonuclear fireballs they could generate?[3] How do quantum effects alter the dynamics of sound propagation? Are we conceiving answers to these questions? And if not, are we conceiving anything at all? I do not think there are good answers here, and I do not think we should be confident in our ability to conceive or imagine truly outlandish situations. Yet Lewin's theory requires exactly that.

Reason 2. Conceivability is not necessary to explain ordinary practice.

I also doubt that we need to try to conceive these outlandish possibilities. For Lewin's entire line of thinking arguably rests on a simple mistake: Infinite group structure inheres in abstract entities rather than concrete particulars, numbers rather than pitches or temperatures. The group of real numbers is an abstract mathematical object that usefully models physical quantities, allowing us to apply the same operations in many different circumstances (e.g., "π semitones below middle C," "π degrees below the boiling point of water"). But we can typically do so only within a certain range of validity: Terms like "10 degrees colder than absolute zero" or "5000 miles south of the Tropic of Capricorn" are simply undefined, because the mapping between numbers and physical states is only partial.[4] This is not a defect to be remedied by heroic acts of imagination; rather, it reflects the fact that the infinite group structure belongs to our mathematical *models* rather than the things we apply them to.[5] I am somewhat sympathetic to the claim that we can, in an appropriately idealized sense, conceive the sum of any two numbers; I am completely unsympathetic to the claim that we can conceive of altering every *physical* quantity by every numerical amount.

Note that it is technically possible to define the action of an infinite group on a finite set. Figure 1.1a models the finite space of pitches playable on a standard piano. Instead of postulating an infinite space of physical pitches, we can define a virtual pitch as a pair (p, o) where o is an octave number and p is the physical pitch whose octave is closest to o; thus the virtual pitches with pitch class C are pairs of the form . . . (C7, 7), (C8, 8), (C8, 9), (C8, 10), ... We can then define an action of the octave shifts that operates normally on physical pitches,

2 Thanks here to Mark Newman.

3 Monroe (2014) imagines the effects of a baseball thrown at near light speed; compare Lewin (1987) 2007: 30.

4 See Teh 2024: sec. 2.1, esp. 10, ("modelling space as $\mathbb{R}^3$ is not to represent it as infinite [as a confused literalistic reading might suppose]") and 15 ("the mathematical medium of a physical representation should not be literalistically interpreted as suggesting some particular ontology").

5 To be sure, finite group structure can often be said to inhere in concrete particulars.

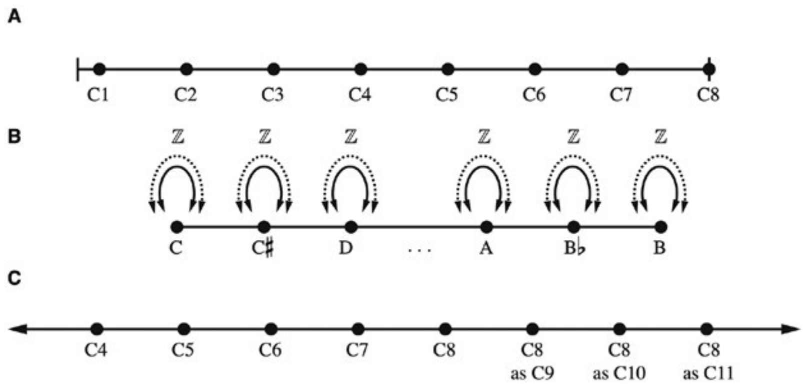

Figure 1.1. A. The bounded space of the piano keyboard. B. The quotient space that results when octave shifts act on virtual pitches; dotted arrows symbolize inaudible and merely conceptual octave shifts. C. The quotient space's universal covering groupoid contains points that represent pitches *understood in a certain way*, such as "C8 considered as the result of transposing C8 up by octave."

but sends (C8, 8) to (C8, 9). (That is, the operation "transpose up by octave," when applied to the highest C on the piano, simply leaves that note unchanged, rather than transposing it off the edge of the keyboard; this is what often happens in actual music.) Standard mathematics allows us to form the quotient shown in Figure 1.1b; this is the pitch-class circle, containing an infinite number of octave transpositions at every point, most of which have no audible effect, since they lie beyond the edge of the keyboard. We can even form this space's "universal covering groupoid," an infinite line containing pitches *considered in a certain way*: "C8 considered as C9" and "C8 considered as C10," and so on. (Transposing "C8 considered as C10" up by octave produces "C8 considered as C11"; transposing it down by octave produces "C8 considered as C9.") This is a well-defined action of the infinite group of octave transpositions on the space of virtual pitches.

It is also a mathematical contrivance rather than a tool for describing human behavior—much less a general answer to the question, How does infinite mathematics apply to a finite world? For our model assigns determinate meaning to phrases like "the C that is 10000 octaves above middle C" ("C8 considered as C10004") whereas ordinary musical discourse does not. Similarly, the analogous temperature model assigns meaning to phrases like "10 degrees colder than absolute zero," which are typically considered meaningless. Mathematically, it is interesting to consider the action of an infinite group on a finite set. Philosophically, we need to recognize that infinite mathematics provides a useful yet limited model of the actual world.

Reason 3: Conceivability is not a reliable guide to actuality.

I can conceive a chess move in which the queen proceeds horizontally and then diagonally, but this is not part of the game that humans actually play. What matters, in thinking about chess, are *the moves that are used* rather than the moves we can conceive. Similarly, I can imagine producing a tempo ratio of π by moving my hands at the same speed along two different shapes—say, back and forth along a line segment of length x, and around a circle with diameter x. But it is impossible to move steadily enough to accurately approximate π in this way.

Why did I try to imagine that last possibility? Because David Lewin actually proposed it (Lewin [1987] 2007: 67). Here it is useful to remember that Lewin was a twelve-tone composer whose emphasis on conceivability arose from his utopian aesthetics. Like many of his fellow Babbitt students, he was less interested in perception and practicality than in describing abstract musical structures. Lewin proposed his unworkable "exercise in conducting the tempo ratio of π" despite knowing that it was impractical, and despite knowing that each and every one of his readers was in a position to verify that for themselves. This is remarkable, as it suggests an entire intellectual milieu that had largely lost touch with questions of practicality. Evidently, Lewin was confident that none of his readers would say, "Hey, that's not a good way to approximate π!"—much as he was confident they would not think too hard about what "transposition by a trillion semitones" involved, or whether operations like $XY\mathbf{T}_4$ felt uniform.

The strategies we have considered are all united by the thought that abstract properties such as definability or conceivability are good guides to musical significance. In this respect they reflect the legacy of utopian modernism: Like twelve-tone theory, they deemphasize human limitation, directing attention to formal systems rather than the particularities of psychology, perception, and practice.[6] Widespread acceptance of transformational closure suggests that contemporary music theory has not come to grips with its modernist inheritance. I think there is something inherently problematic about melodic transformations that are completely insensitive to octave information (e.g., Figure 0.1), a radical abstraction that originates with Schoenberg's practice. Similarly problematic is the very notion of a set-class, the idea that major and minor triads are absolutely similar to one another ("consonant triads") and absolutely different from every other sonority. (Harmonic consistency is often a matter of degree, with similar-sounding set-classes playing similar musical roles.) I am also suspicious of *Schritts* that transpose major and minor triads in opposite directions, esoteric transformations that play only a minor role in actual music making. Theorists have been slow to acknowledge these issues, in large part because we inhabit a culture that marginalizes practicality and relevance.

2. Spaces, Symmetries, and Intervals

There are some general mathematical reasons to think that music typically presents an unsystematic multitude of transformations, only some of which naturally

6 A similar point applies to theorists who consider *idealized* rather than actual listeners (see Tymoczko 2020a, responding to Fred Lerdahl). Idealization, like the focus on conceivability, can obscure difficult questions about flesh-and-blood human beings.

form groups. To explain this, I will need to summarize an ongoing research project that uses category theory to unite transformational theory with voice-leading geometry. Though it is not possible to describe that project in detail, I can illustrate its main points with a simple example. This will allow us to distinguish three different kinds of transformation: symmetries, like transposition and inversion; intervals, like the neo-Riemannian transformations; and ad hoc transformations belonging to neither category. This last group includes the contrapuntal interchanges to be defined in Section 4.

I define a *transformational system* as a combination of two elements, a *space* and a group of *symmetries* acting on that space.[7] The space connects points or objects with paths or arrows. Symmetries are invertible transformations that leave the space "essentially unchanged"; they form equivalence classes of both objects and paths. Equivalence classes of paths constitute a second set of transformations known as *intervals*. *Loops* are intervals connecting objects in the same equivalence class; they have a natural group structure and are typically isomorphic but not identical to the symmetries. Many familiar musical transformations, including voice exchanges and neo-Riemannian transformations, are loops. The intervals connecting different equivalence classes do not have group structure as they are not necessarily composable: An interval $a \rightarrow b$ does not in general combine with an interval $c \rightarrow d$. A transformational system can also contain ad hoc transformations, equivalence classes of paths not associated with symmetries; once again, these need not have any group structure at all.[8]

Figure 2.1 depicts a twelve-object space representing three percussionists who either play agogo, bongos, and claves, or drum, egg shaker, and flexatone. Order corresponds to performer, so that *abc* indicates that percussionist 1 plays agogo, percussionist 2 plays bongos, and percussionist 3 plays claves; *fde* indicates that percussionists 1, 2, and 3 play flexatone, drum, and egg shaker, respectively. So far, this is a space rather than a transformational system: If we move from *abc* to *def* we do not apply a general transformation defined at other points in the space; we simply move from one state to another. We can travel between distant states by composing paths: Moving from *abc* to *bac* to *edf* takes us from a situation in which percussionists play agogo, bongos, and claves, to one in which they play egg shaker, drum, and flexatone; conceptually, this is the arrow $abc \rightarrow edf$. (One can imagine this as a direct connection between those objects, not passing through any other object; in principle, the space has 144

7 Mathematically a space is a *groupoid*, or a category in which every arrow is invertible; a symmetry is a groupoid isomorphism, an invertible *functor* from the groupoid to itself. (Technically, I consider *coordinatized spaces*, or mappings between a "space of elements" and an isomorphic "space of attributes.") See Goldblatt 2006 and Leinster 2014 for good introductions to category theory.

8 This formalism is closely related to that in Lewin (1987) 2007, whose "interval-preserving operations" and "canonic group" both correspond to my "symmetry group." Unlike Lewin, however, I distinguish a space from a symmetry group; furthermore, I do not require that symmetries be functions over the objects in a space, nor that they act simply transitively on objects; this eventually leads to substantial differences.

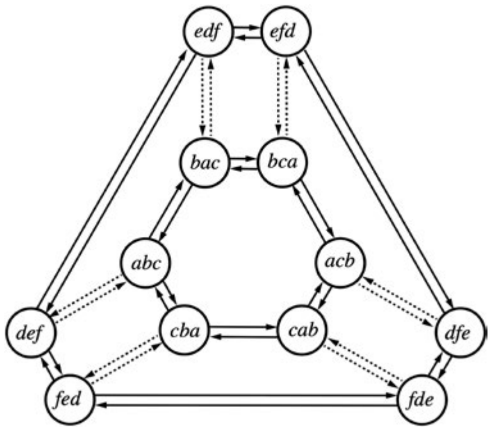

Figure 2.1. A musical space containing the six permutations of two different collections of instruments, *abc* and *def*.

distinct arrows, one for every ordered pair of objects, with arrows combining as one would expect.[9]) This allows us to move around the diagram in something like the way one moves through physical space.

A *symmetry* is an invertible transformation from a space to itself, mapping objects to objects, arrows to arrows, and preserving arrow composition. Since symmetries are invertible, they combine to form groups; in fact, modeling symmetries is one of group theory's central functions. Figure 2.2 defines three symmetries corresponding to reflection around the three lines bisecting the hexagons' opposite edges. Musically, these generate pairwise instrument swaps: Vertical reflection exchanges the instruments played by percussionists 2 and 3, sending *edf* to *efd* and *abc* to *acb*; reflection around the NW/SE axis exchanges the instruments played by percussionists 1 and 2. (Performer permutations are symmetries because it is arbitrary which person we call "performer 1" and which person we put first in our ordering.) Two reflections combine to form a 120° rotation, which are circular permutations like *abc* → *cba*. Thus we have a six-element symmetry group: Geometrically it contains three reflections and three rotations, musically, six performer-based permutations of instruments.

A symmetry group creates equivalence classes of objects related by the group: In our case, there are two equivalence classes, one containing orderings of *a*, *b*, and *c*, and the other *d*, *e*, and *f*. (This is because no symmetry relates any *abc*-object to any *def*-object or vice versa.) Symmetries also create *intervals*, or equivalence classes of arrows related by symmetries: For instance, the symmetry "rotate clockwise by 120°" transforms *bac* → *bca* into *acb* → *cab*, *acb* → *cab* and *cba*

9 Typically, when drawing a space one includes only those arrows necessary to generate the entire collection.

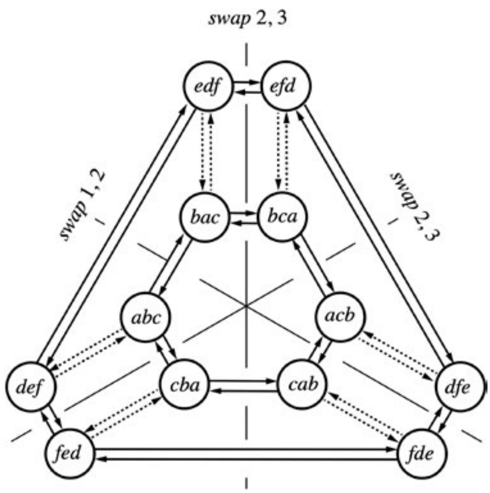

Figure 2.2. Reflection around the three axes of symmetry swaps the instruments played by two percussionists. For instance, vertical reflection swaps the instruments played by percussionists 2 and 3, sending *edf* to *efd*, *bac* to *bca*, *abc* to *acb*, and so on.

→ *abc,* and *cba* → *abc* into *bac* → *bca*. These three arrows belong to the same equivalence class, forming a transformation that permutes the agogo and claves regardless of which percussionist plays them. Music theoretically, it is a "contextual" rather than "absolute" permutation, defined relative to *instruments* rather than percussionists.

A little investigation will reveal that equivalence classes of arrows related by symmetries *always* have a contextual character, being instrument based rather than percussionist based: Intervals have the form "switch instrument *x* with instrument *y*" whereas symmetries have the form "switch the instrument played by percussionist *x* with the instrument played by percussionist *y*."[10] (Geometrically, intervals correspond to congruent paths in Figure 2.1.) This is generally true: For virtually any musical space, symmetries are *absolute*, or content neutral, while intervals are *contextual,* or content dependent. It can be shown that intervals apply symmetry operations in reverse-chronological order.[11]

We can explore these equivalence classes by forming *quotient spaces* that glue together equivalent objects and arrows. Figure 2.3 shows the quotient of our twelve-object space by our six-element symmetry group. The space has two

10 Yust 2019 explores this duality in a closely analogous context.

11 Consider a progression $x \rightarrow \boldsymbol{g}(x)$, with $\boldsymbol{g}$ some element of the symmetry group. An interval is an equivalence class of progressions $\boldsymbol{f}(x) \rightarrow \boldsymbol{fg}(x)$, with $\boldsymbol{f}$ ranging over the entire symmetry group. This equivalence class can be interpreted as a transformation $\boldsymbol{g}$ that sends every $\boldsymbol{f}(x)$ to $\boldsymbol{fg}(x)$, applying $\boldsymbol{g}$ *before* $\boldsymbol{f}$.

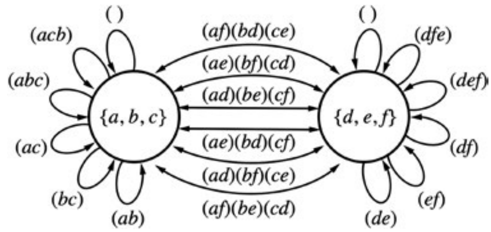

Figure 2.3. The quotient space that results from gluing together objects and arrows related by symmetries in Figure 2.2.

objects corresponding to the two sets of instruments; each is connected to itself by six loops representing the six instrument-based permutations; each object is connected to the other by six instrument-changing arrows. The loops at each object each form a group, known as the object's *automorphism* group. In a typical transformational system, the automorphism groups will be isomorphic both to the symmetry group and to one another. But those isomorphisms involve the groups as a whole rather than their component transformations: For a noncommutative symmetry group, there is no one symmetry corresponding to any particular interval, and no canonical correspondence between the specific intervals at different points.

In other words, this simple transformational system presents three distinct groups: a symmetry group defined on the entire space, and two automorphism groups of loops whose elements are equivalence classes of arrows in the original, twelve-object space. The three groups are isomorphic but not identical, and do not combine to form a larger group, at least not one that is musically or mathematically natural. Musically, this is because operations on one equivalence class, such as "switch agogo and bongos," have no analog on the other. (There is, in other words, no principled reason to associate the agogo with either drum, egg shaker, or flexatone.) Nor is there any reason to connect a specific symmetry such as "switch the instruments played by percussionists 1 and 2" to a specific interval such as "switch bongos and claves." Mathematically, this is because symmetries and intervals are ontologically different, the former being transformations of the entire space, the latter being equivalence classes of paths through the space. All of this is standard mathematics that is neither original to me nor specific to music.[12]

These different groups provide musicians with multiple transformational vocabularies, allowing them to permute either percussionists or instruments.

12 Symmetries are what algebraic topologists call *deck transformations*, while intervals correspond to the action by lifting. These transformations are isomorphic but not canonically so; the same is true for the fundamental groups at different points in a space (Hatcher 2001).

		(12)		(123)		(12)		(123)	
perc 1	*a*		*b*		*c*		*b*		*a*
perc 2	*b*		*a*		*b*		*c*		*b*
perc 3	*c*		*c*		*a*		*a*		*c*
		(*ab*)		(*abc*)		(*bc*)		(*bac*)	

Figure 2.4. Described using percussionist-based permutations, this sequence alternates between two transformations, (12) and (123); described using instrument-based permutations, it uses four separate transformations.

A particular passage might be more systematic according to one or the other description (Figure 2.4). It could also happen that musicians *combine* these different transformations into unified composites (e.g., "swap agogo and bongos and then swap the instruments played by performers 1 and 2"). In that case, the analyst would have two options. One is to model the composite transformation as an ad hoc equivalence class rooted in compositional practice, rather than the symmetry group itself. (As we will see, such transformations are ubiquitous: Musicians use a very wide range of transformations, only some of which derive from symmetries; and symmetries are often combined with intervals.) The other option is to construct a new symmetry group combining the two kinds of permutation. This would not eliminate group-theoretical pluralism, however, as the new symmetry group would give rise to new automorphism groups containing new transformations that composers would again be free to use.

In future work I hope to explore the music-theoretical ramifications of these ideas, generalizing the relation of symmetry and interval, or absolute and contextual transformation, to arbitrary groups and spaces. My ultimate goal is to describe transformational analogues to the central procedures of voice-leading geometry—developing conceptual tools equally suited to finite graphs and continuous spaces. For present purposes, however, the relevant points are, first, that a plurality of groups is the mathematical norm rather than the exception; second, that one would not normally construct a group combining symmetries and intervals, as these are mathematically distinct; and third, that while symmetries and loops naturally have group structure, other transformations do not. Intervals can be defined at different regions of a space, in which case they do not necessarily combine.

The upshot is that it takes judgment to know when to combine transformations into groups. It takes musical judgment to know whether a particular combination of transformations is or is not comprehensible as a coherent operation. And it takes mathematical judgment to determine whether a transformation is a symmetry, an interval, or an ad hoc transformation. The choice of transformation group is determined not just by composers' behavior, but also by our theoretical understanding of space in which they operate: Thus it could be reasonable to model instrument and percussionist permutations with separate groups *even if musicians sometimes combine them*. The issues here are subtle and complex, requiring a careful consideration of many different factors. Transformational theory has generally avoided this complexity, defaulting to a generic presumption that transformations form groups.

3. Voice-Leading Transformations

We now turn to the technical business of defining voice-leading transformations. The logic is exactly the same as in the last section: In both cases, we begin with a space of objects and arrows, with no intrinsic transformational content; we then define symmetries acting on that space. These symmetries are themselves transformations (like "swap the instruments played by percussionists 1 and 2"), but they also generate a dual set of intervals, consisting of equivalence classes of arrows related by symmetries ("swap agogo and bongos"). The difference is that we will now be working with an infinite set of objects related by an infinite set of symmetries. In principle, however, everything proceeds as before: Voice-leading transformations are just the intervals corresponding to the familiar "**OPTI** symmetries" of voice-leading geometry (Callender, Quinn, and Tymoczko 2008). Geometrically, they are paths through various voice-leading spaces. Algebraically, they apply the **OPTI** symmetries in reverse-chronological order.

I define a *progression* as a concrete musical event moving from one *n*-note pitch sequence to another (Tymoczko 2011b). Progressions can be represented as pairs of lists of pitches, written (C4, E♭4, G4) → (D♭4, E♭4, G4), and mapping pitch 1 in the first list to pitch 1 in the second (C4 to D♭4), pitch 2 to pitch 2 (E♭4 to E♭4), and so on. They correspond to phrases like "C4 in the first instrument moves to D♭4 in that same instrument, while E♭ remains fixed in the second instrument, and G4 remains fixed in the third." Geometrically they are arrows in ordered pitch space.

Voice leadings are the intervals associated with the **OP** symmetries, equivalence classes of progressions related by octave shifts and permutation; they are written (C, E♭, G) $\xrightarrow{1,0,0}$ (D♭, E♭, G) and correspond to phrases like "C in any instrument or octave moves up by semitone to D♭, while E♭ and G remain fixed." I omit the numbers above the arrow when all notes move by the shortest possible distance; thus the preceding voice leading can also be written (C, E♭, G) → (D♭, E♭, G). A *path in pitch-class space* is a one-voice voice leading such as C → D♭, which moves C in any octave to the D♭ one semitone above it.

A *transpositional voice-leading schema* is an interval of the **OPT** symmetry group, an equivalence class of progressions related by octave shifts, permutations, and transposition; these correspond to phrases like "raise the root of the minor triad by semitone, holding the other two notes fixed" (Tymoczko 2020b).

An *inversional voice-leading schema* is an interval of the **OPTI** symmetry group, adding inversion to the mix ("either semitonally raise the root of a minor triad or semitonally lower the fifth of a major triad").

The **OPTI** symmetries are rigid operations that either preserve paths in pitch-class space (**O**, **P**, **T**) or multiply them by −1 (**I**). As a result, their corresponding intervals move their voices by the same collection of distances. Rigid transformations are important, in part, because they can be assigned a consistent voice-leading size: The transpositional voice-leading schema (C4, E♭4, G4) → (D♭4, E♭4, G4) always moves just one note by just one semitone. By contrast, the transformation $XY\mathbf{T}_4$ in Figure 0.3 moves its voices along different paths, which makes it harder to assign it a voice-leading size, and harder to perceive "$XY\mathbf{T}_4$ as applied to the minor triad" as the same as "$XY\mathbf{T}_4$ as applied to the 026 trichord." To be sure, we can sometimes perceive unity in nonrigid transformations. But rigid transformations are generally more straightforward than their nonrigid counterparts.

Many voice-leading transformations have familiar names. A *voice exchange* is a transpositional voice-leading schema that is equal to its retrograde; it can contain pairs of retrograde-related paths $\{x \xrightarrow{i} y, y \xrightarrow{-i} x\}$, the first mapping pitch-class x to pitch class y by i semitones, the second mapping pitch class y to pitch class x by $-i$ semitones. These are *pairwise voice exchanges* that swap two pitch classes along equal and opposite paths. A voice exchange can also contain common tones of the form $x \xrightarrow{0} x$ since these are retrograde-invariant. For major and minor triads, we can notate pairwise voice crossings as "c_{RT}," where "RT" indicates the triadic elements to be crossed (in this case, root and third). A chord's pairwise voice exchanges generate the *voice-crossing subgroup*, consisting of voice leadings $X \rightarrow X$ whose paths collectively sum to 0. The factoring-out of voice exchanges plays an important role in both Schenkerian analysis and voice-leading geometry (Tymoczko 2023c).

A *spacing-preserving* transformation preserves spacing as measured along the *intrinsic scale* formed by a chord's own notes (Figure 3.1). I have described these as "strongly crossing free," since their voices will never cross no matter how they are arranged in register. Spacing-preserving voice leadings are what remain when one ignores ("quotients out" or eliminates) voice exchanges; they combine with the voice exchanges to produce *all* the bijective voice leadings between any two chords. (Geometrically, voice exchanges are represented by paths that bounce off mirror singularities, while spacing-preserving voice leadings avoid those singularities [Tymoczko 2020b].) Since voice exchanges never make a voice leading smaller, there is always a minimal voice leading that is spacing preserving (Tymoczko 2011b). A *transposition along the chord* is a spacing-preserving voice leading from a chord to itself; these form the group of registral inversions. We can notate transpositions along the chord "t_x," with x the number of steps along the intrinsic scale.

Figure 3.1. The transpositional voice-leading schema Y: (C, E♭, G) → (B, F, G) preserves spacing as measured in steps along the *intrinsic scale*, the octave-repeating scale formed from a chord's own notes. The intrinsic scale is shown on the lower staff, with the chordal notes in open noteheads.

A *contextual inversion* is an inversional voice-leading schema whose inversion is equal to its retrograde—or equivalently, an inversional voice-leading schema that is retrograde-inversion invariant. These can be decomposed into retrograde-inversionally related voice pairs $\{x \xrightarrow{i} (c-y), y \xrightarrow{i} (c-x)\}$, and single voices connecting inversionally related notes $\{x \xrightarrow{i} (c-x)\}$. Contextual inversions include "inversional" voice leadings mapping each note to its inversional partner, such as (C, E, G) $\xrightarrow{7,-1,-7}$ (G, E♭, C), as well as the parsimonious voice leadings of neo-Riemannian theory.

The earliest neo-Riemannian theorists treated $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{R}$ as *purely harmonic* operations that did not imply any particular mapping between their notes (Lewin [1987] 2007; Hyer 1989); other theorists interpreted these transformations as contextual inversions mapping each voice to its inversional partner (Morris 1998; Straus 2003). In 1996, Richard Cohn described the transformations in yet another way, as *parsimonious voice leadings* with two common tones and one voice moving by step (Cohn 1996, 1997).[13] We can generalize Cohn's idea by defining a neo-Riemannian voice leading as a spacing-preserving contextual inversion; this extends the neo-Riemannian transformations to any set-class whatsoever (Figure 3.2).[14] Spacing-preserving voice leadings between inversionally related chords preserve the distance between at least two voices.

Voice-leading transformations provide useful labels for contrapuntal moves: Given a suitable corpus, one can, for example, survey all the different transpositional voice-leading schemas used to express a particular harmonic progression (Figure 3.3). We use transpositional voice-leading schemas when we teach our students "resolutions of *the* vii°6 chord," treating vii°6 as a general chord type rather than a particular set of notes. They are also widely used in music

13 Since Cohn used traditional pitch-class intervals, his "$\mathcal{R}$ transform" does not distinguish between Figures 0.1c–d, whereas we consider them to be different voice-leading schemas.

14 Tymoczko 2023b notes that these voice leadings can be understood as *dual inversions* that invert both in the intrinsic scale and in an extrinsic or enclosing scale.

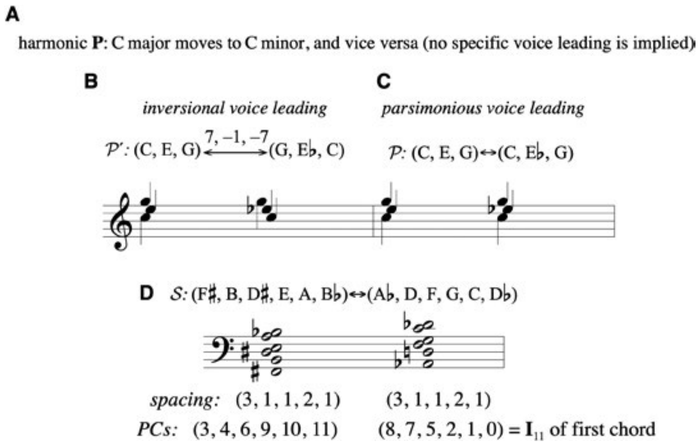

Figure 3.2. The "parallel" transform as a purely harmonic transformation (A); as an inversional voice leading (B); as a parsimonious voice leading (C). In D, a spacing-preserving contextual inversion from the opening of Schoenberg's *Violin Concerto*, which can be considered a generalized Cohnian voice leading. The spacing is listed in intrinsic steps, along the notes of each chord.

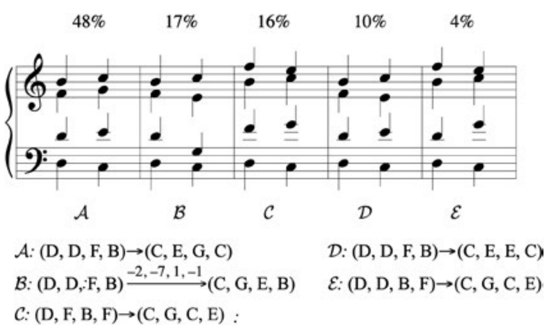

Figure 3.3. The most-common resolutions of vii°6–I in Bach's chorales, expressed as transpositional voice-leading schemas.

software: music21, for example, interprets neo-Riemannian operations as transpositional voice-leading schemas moving voices by short distances.

Sequences repeatedly apply a single voice-leading transformation. Figure 3.4 shows a passage from Palestrina in which the upper voices repeat a single transpositional voice-leading schema, with the repetition somewhat disguised by nonsequential rhythm and non-harmonic tones. (I have argued that these sorts of "repeating contrapuntal patterns" play an important role in the Renaissance prehistory of the sequence [Tymoczko 2023c]). Figure 3.5 shows a more obviously sequential passage from Bach. It also embellishes a single transpositional

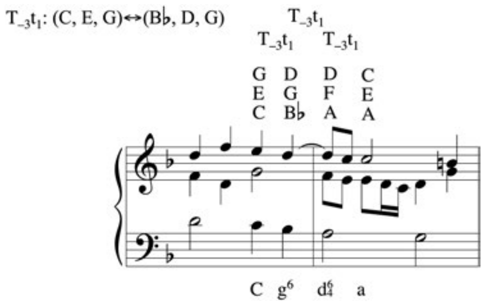

Figure 3.4. A voice-leading sequence in the Kyrie of Palestrina's *Missa descendit angelus Domini*, mm. 39–40. The diatonic transpositional voice-leading schema $T_{-3}t_1$ appears three times in a row, producing four chords related by ascending fifth.

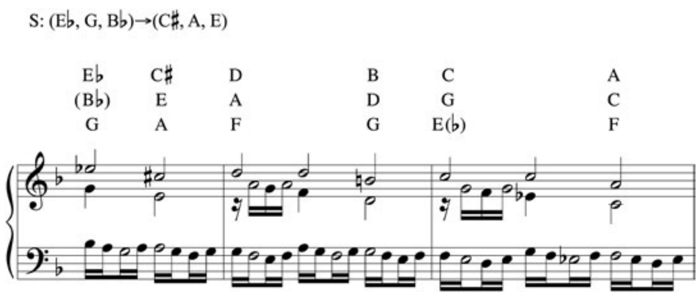

Figure 3.5 Measures 9–11 of *The Well-Tempered Clavier*'s first D-minor fugue present a lightly-disguised diatonic transpositional voice-leading schema. The outer voices move in canon, exchanging root and third, while the middle voice always sounds the fifth. The schema is not exact, as the middle voice alternates ascending fourth and descending fifth, rather than moving consistently by one interval. The underlying regularity is also disguised by the figuration, which repeats every two chords rather than every chord, obscuring the canonic relation between outer voices.

voice-leading schema: first, by alternating ascending fourth and descending fifth in the middle voice;[15] second, by using figuration that repeats every two chords rather than every chord; and third, by obscuring the canonic repetition of the outer voices.

As intervals, voice-leading transformations form groups that are isomorphic but not identical to their associated symmetries: The group of voice leadings from a chord to itself is isomorphic to the group of octave shifts and permutations; while the group of transpositional voice-leading schemas, connecting

15 Literal repetition would be awkward registrally.

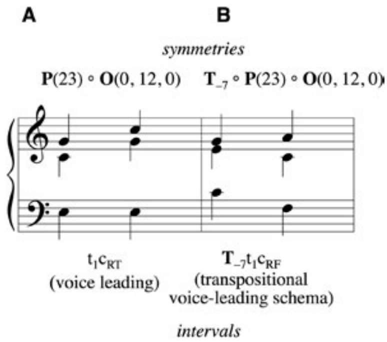

Figure 3.6. Musical progressions can generally be described either with symmetries (top) or intervals (bottom). Voices are labeled bottom-to-top. In A, the progression can either be described with the symmetry "displace the middle voice up by octave then swap voices 2 and 3" or by the interval "cross root and third and then transpose up by one triadic step." In B, the progression can either be described with the symmetry "displace the middle voice up by octave then swap voices 2 and 3 then transpose down by perfect fifth" or by the interval "cross root and fifth, transpose up by one triadic step, and then transpose down by perfect fifth."

a transpositional set-class to itself, is isomorphic to the group of octave shifts, permutations, and transpositions. This provides two equivalent vocabularies for describing progressions (Figure 3.6). I have argued that the notion of a "chord" arose out of a gradual shift from thinking with symmetries to thinking with intervals, a process that could be described as a collective, unconscious, and centuries-long group-theory calculation (Tymoczko 2023c: chap. 6). Voice-leading transformations do not have group structure when they connect different chords, but that is to be expected: Mathematically, they are paths in a space, and only loops-at-a-point typically form groups; paths between distinct points do not.

All of this makes for elegant mathematics but awkward music theory, as there is no one symmetry group that is completely satisfactory analytically. The problem is that no one species of voice-leading transformation contains both contextual inversions and ordinary transpositions. There are transpositional voice-leading schemas that act like transpositions, since transpositions are both symmetries and intervals of transpositional set-class space. But there are no inversional voice-leading schemas that act like transpositions, as inversional voice-leading schemas necessarily move inversions in opposite directions; transpositions are symmetries but *not* intervals in inversional set-class space. Contextual inversions, meanwhile, are inversional voice-leading schemas; there is no transpositional voice-leading schema, and hence no interval in transpositional set-class space, that acts like a contextual inversion. If we want to combine transposition with contextual inversion, as in Figure 0.2a–c, we therefore need to combine intervals from different symmetry groups; these combinations are *not* voice-leading transformations as I have defined them. This is mathematically and conceptually awkward. It is analogous to the dilemma faced at the end of Section

2, when we contemplated the transformation "swap agogo and bongos and then swap the instruments played by performers 1 and 2."

That dilemma was the original motivation for this article. Personally, I find contextual inversions and ordinary transpositions to be useful; furthermore, I think composers sometimes combine them to good effect, as in Figures 0.2a–c. How can we model these composites? Traditional theorists have either limited themselves to inversional contrapuntal interchanges or constructed new symmetry groups combining both symmetries and intervals. For reasons to be discussed in Sections 7–8, I find these solutions to be unsatisfactory. But if voice-leading transformations are not the answer, then what is?

4. Contrapuntal Interchanges

Contrapuntal interchanges use the retrograde to form equivalence classes of voice-leading transformations, combining $X \rightarrow Y$ and $Y \rightarrow X$ into a single composite $X \leftrightarrow Y$. Contrapuntal interchanges are not voice-leading transformations, as the retrograde is not a symmetry operation.[16] But they resemble voice-leading transformations in many ways, being equivalence classes of progressions and rigid transformations that multiply voice-leading paths by −1.

A *transpositional contrapuntal interchange*, or just *contrapuntal interchange*, can be written (C, E♭, G) ↔ (D♭, E♭, G); it corresponds to a phrase like "either semitonally raise the root of the minor triad, or semitonally lower the 0 of a 026 trichord." An *inversional contrapuntal interchange* combines an inversional voice-leading schema with retrograding and corresponds to a phrase like "either semitonally raise the root of a minor triad or semitonally lower the fifth of a major triad or semitonally lower the 0 of a 026 trichord or semitonally raise the 6 of a 046 trichord." Thus we have the hierarchy of equivalence classes shown in Figure 4.1, each grouping progressions in different ways. We can identify any of these transformations by providing a specific example (a progression) along with the relevant collection of operations (generating an equivalence class). This is one reason that they are so useful: A single example, coupled with knowledge of the relevant operations, suffices to define the transformation itself; in this sense, they are musically *uniform*, doing "the same thing" to every object in their domain.

Contrapuntal interchanges are most directly useful when analyzing passages that alternate between two different sonorities, whether set-theoretically related or not. The transformations $\mathcal{X}$: (C, E♭, G) ↔ (D♭, E♭, G) and $\mathcal{Y}$: (C, E♭, G) ↔ (B, F, G) connect minor triads and 026 trichords. Figure 4.2 shows them at the start of the development of the first movement of Beethoven's op. 59, no. 2; Figure 4.3 shows $\mathcal{X}$ in three passages from Stravinsky's early ballets. On this analysis, Figure 4.3a is a *sequence* that repeatedly applies the single transformation $\mathcal{X}\mathbf{T}_2$.

16 The retrograde allows us to form equivalence classes of arrows but not objects.

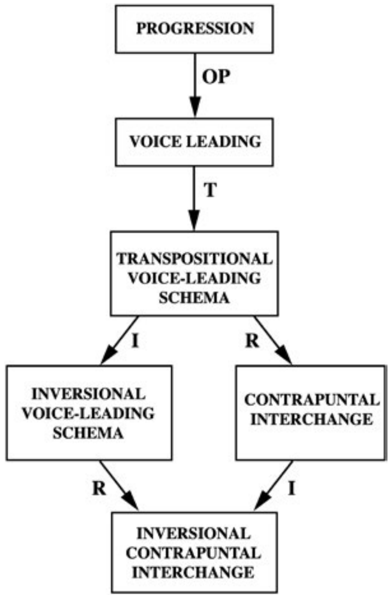

Figure 4.1. Equivalence classes and the operations forming them. To move from a progression to a voice leading, form an equivalence class containing octave- and permutation-related progressions; to produce a transpositional voice-leading schema form an equivalence class of voice leadings related by transposition.

In general, contrapuntal interchanges allow us to interpret oscillating harmony as the product of a repeating transformation.

We previously defined a (Cohnian) "neo-Riemannian voice leading" as a spacing-preserving contextual inversion. Contrapuntal interchanges are even-more-generalized Cohnian transformations: Like voice-leading transformations, they imply specific contrapuntal moves; and like Cohn's transformations, they are involutions, sending chord *A* to *B*, and *B* back to *A*. Unlike Cohn's transformations, contrapuntal interchanges need not connect triads or even inversionally related sonorities, nor need they involve any particular kind of voice leading (e.g., "parsimonious" or "spacing-preserving"). Instead they can be constructed from *any voice leading whatsoever*. In this respect they represent a comprehensive generalization of Cohn's idea.

This generalization involves a substantial philosophical shift, however. Neo-Riemannian theory conceives its basic transformations as *inversions*; in my view, this fully justifies the adjective "Riemannian," despite that theory's many departures from Hugo Riemann. I am trying to avoid the symmetry-centered modernism inherent in this approach; contrapuntal interchanges thus reinterpret contextual inversions using the *retrograde* relationship. As we will discuss in Section 6, this deemphasizing of inversion is part and parcel of a more general move away from set-classes: I find it limiting to conceive harmonic categories as isolated monadnocks completely unrelated to one another; instead, I prefer to think about degrees of resemblance, replacing the dichotomous notion of set-class with a fuzzier concept of similarity (Quinn 2001, 2006, 2007; Tymoczko 2023a). In particular, I do not like to think of major and minor triads as uniquely consonant

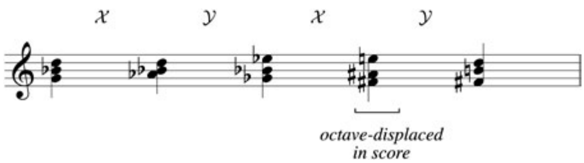

Figure 4.2. The *X* and *Y* transforms at the start of the first-movement development of Beethoven's op. 59 no. 2 (mm. 69ff).

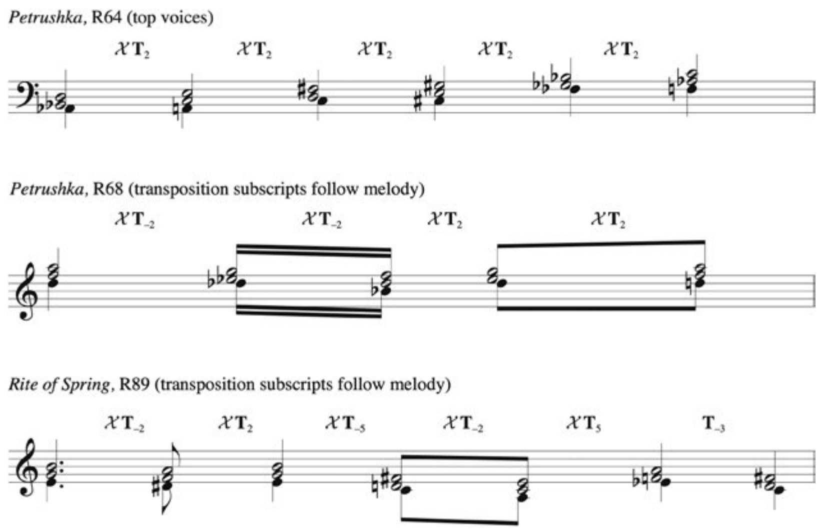

Figure 4.3. Three passages in which Stravinsky uses the *X* transform.

objects unrelated to other species of harmony. Contrapuntal interchanges thus generalize contextual inversions so that they can link chords in different set-classes. They are one aspect of a larger project of identifying alternatives to traditional theoretical notions such as inversion, set-class, and pitch-class interval.

An even number of *L*, *P*, and *R* transformations is not a contextual inversion but a dualist Schritt transposing major triads in one direction and minor triads in the other. The same is true for contrapuntal interchanges more generally. Suppose we have a pair of involutions X: $A \leftrightarrow B$ and Y: $B \leftrightarrow \mathbf{T}_x(A)$ that commute with transposition; then the combination XY will be a Schritt that sends A to $\mathbf{T}_x(A)$ and B to $\mathbf{T}_{-x}(B)$. Figure 4.4 provides the abstract schema while Figure 4.5 illustrates it with X equal to (C, E♭, G) ↔ (D♭, E♭, G) and Y equal to (C, E♭, G) ↔ (B, F, G) (cf. Figure 0.3). Not only does XY transpose the minor triad down by four semitones and the 026 trichord up by four semitones, but it also operates oppositely on the chord's individual notes: XY moves the root of the minor triad *up* to the third of the minor triad (and third to fifth and fifth to root); labeling the 026 trichord as an incomplete seventh, it sends third *down* to the root (and root to

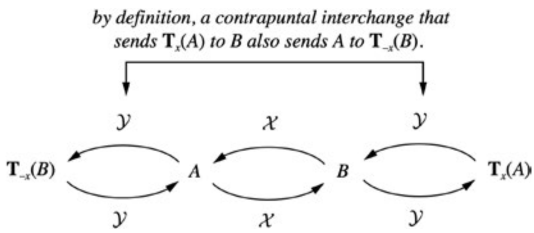

Figure 4.4. Contrapuntal interchanges generally combine to form dualist transformations.

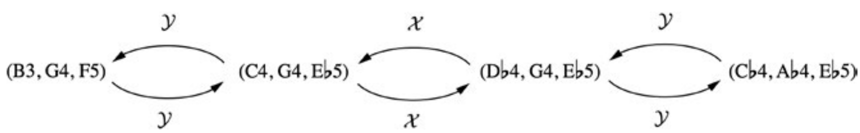

Figure 4.5. Dualist transformations connecting minor triads to 026 trichords. The contrapuntal interchanges *X*: (C, E♭, G) ↔ (D♭, E♭, G) and *Y*: (C, E♭, G)↔ (B, F, G) combine to produce a "doubly dualist" Schritt. When applied to minor triads *XY* produces a descending-major-third root progression that moves notes up by one chordal step (root to third, third to fifth, fifth to root); when applied to 026 trichords it is an ascending-major-third root progression that moves notes down by chordal step (root to seventh, seventh to third, third to root).

seventh and seventh to third). We can say that combination *XY* is *doubly dualist*, moving chords in opposite directions both chromatically and along the chord's intrinsic scale: In the terminology of Section 6, it acts as $\mathbf{T}_{-4}\mathrm{t}_1$ on minor triads and $\mathbf{T}_4\mathrm{t}_{-1}$ on 026 trichords. In traditional neo-Riemannian theory, this dualism is associated with the inversional relationship between major and minor. Contrapuntal interchanges give us dualism without inversion, as the simple consequence of traversing the same arrows in opposite directions.[17]

Mathematically, this means that two contrapuntal interchanges do not necessarily compose to form a contrapuntal interchange—instead, they can compose to form a Schritt, which is a different kind of transformation. By contrast, two composable voice-leading transformations *always* produce another voice-leading transformation of that type. (This a consequence of the fact that voice-leading transformations are paths through a space, or arrows in a groupoid.) The principle of transformational closure thus requires us either to accept Schritts into our transformational universe or keep contrapuntal interchanges out of it. I think contrapuntal interchanges are useful, whereas Schritts generally are not. I therefore need to abandon transformational closure.

17 Henry Klumpenhouwer (1994: para. 5, qtd. in 2000: 162) suggests that involutions generally combine to produce Schritts; this is not true, since absolute inversions combine to form transpositions. One needs the additional assumption that each involution $A \leftrightarrow B$ commutes with ordinary transposition, sending $\mathbf{T}_x(A)$ to $\mathbf{T}_x(B)$, and $\mathbf{T}_x(B)$ to $\mathbf{T}_x(A)$ for all transpositions $\mathbf{T}_x$.

5. Singularities and Functions

One of my ongoing theoretical projects is to unify the theory of voice-leading with transformational theory. This work is inspired by algebraic topology, which gets its name because it uses groups (algebraic objects) to model loops-through-a-space (geometrical objects); musically, this corresponds to using functions (group elements) to model musical events (paths in some musical space). *Singularities*, or chords that are invariant under a symmetry operation, are a major obstacle here. For when chords are nonsymmetrical, voice-leading transformations and contrapuntal interchanges can be associated with functions from notes to paths in pitch-class space, which in turn determine new chords.[18] When applied to symmetrical chords, however, these operations become ambiguous. Consider the transpositional voice-leading schema S: (C, E, G♯) → (C, E, A). This applies to {C, E, G♯} in three different ways, each raising a different note by semitone.[19] From a voice-leading perspective this multivalence is not necessarily a problem, as we can always specify which version of the transformation we have in mind. If we want to model transformation as functions, then we do not have this option.

Mathematicians have devised various tools for using functions in the presence of singularities (e.g., the "orbifold fundamental group," discussed in Hughes 2015 and 2022). Musically, the issues are largely, though not entirely, notational. Conceptually (C, E, G♯) → (C, E, A), (C, E, G♯) → (C, F, A♭), and (C, E, G♯) → (C♯, E, G♯) are all instances of the same voice-leading transformation. Our question is whether we can define a function that distinguishes these different alternatives. One option is to consider chords to be embedded in continuous pitch-class space. This allows us to "desymmetrize" them by detuning their notes ever so slightly—for example, replacing the symmetrical diminished triad {0, 4, 8} with the aurally indistinguishable but nonsymmetrical {0, 4, 8 – ϵ}, where ϵ is some very small parameter, greater than zero but small enough to be musically inconsequential (say, one thousandth of a semitone). We can then define our transpositional voice-leading schema as S: (0, 4, 8 – ϵ) → (0, 4, 9). This distinguishes the root-position augmented triad {0, 4, 8 – ϵ} from the first-inversion triad (0, 4 – ϵ, 8), notationally if not aurally. (Such detuning is possible even if we are working inside a nonchromatic scale such as the diatonic [Tymoczko 2011b: chap. 4].) The advantage of this approach is that it does not require us to change any of our basic definitions.

An alternative strategy desymmetrizes chords using *attributes* and *elements*. In this framework *attributes* are (potentially arbitrary) labels such as "root," "third," and "fifth," and musical objects are maps between attributes and elements, so that a point in a musical space is a set of (attribute, element) pairs. This allows us to define distinct augmented triads, {(root, C), (third, E), (fifth, G♯)},

18 These transformations are functions, but not in the sense of traditional transformational theory, because they map objects of one kind (a pitch class) to those of another (a path in pitch-class space).

19 Ambiguous transformations can sometimes be modeled as relations (Tymoczko 2009; Hook 2023: sec. 2.4).

{(root, E), (third, G♯), (fifth, C)}, and {(root, G♯), (third, C), (fifth, E)}. These assignments, like the previous paragraph's detunings, provide *interpretations* of a chord, singling out one particular note as the "root."[20] In the case of a nonsymmetrical chord, interpretation is unnecessary, as we can infer the attributes from the elements; for a symmetrical chord a single set of elements supports multiple attribute assignments, or interpretations. Philosophically, I find attributes preferable to detuning. Unfortunately, they require a more comprehensive reformulation of standard music theory; for simplicity, I will therefore use detuning in what follows.[21]

Desymmetrization links voice-leading geometry with transformational theory, providing a functional notation applicable even in the presence of singularities. I will use roman font for transpositional and inversional voice-leading schemas and cursive letters for contrapuntal interchanges. Let $\mathcal{S}$: (C, E, G♯ – ϵ) ↔ (C, E, A) be the contrapuntal interchange that moves one note of the augmented triad up by one semitone and $\mathcal{R}$: (C, E, G) ↔ (C, E, A) be the neo-Riemannian "relative" voice leading. Then $\mathcal{RS}/\mathcal{SR}$ will be the contrapuntal interchange (C, E, G♯ – ϵ) ↔ (C, E, G), which applies $\mathcal{R}$-then-$\mathcal{S}$ to a major triad and $\mathcal{S}$-then-$\mathcal{R}$ to an augmented triad. Any single-semitone voice-leading involving an augmented triad and either a major or minor triad can be uniquely labeled as an instance of either $\mathcal{S}$ or $\mathcal{RS}/\mathcal{SR}$; this in turn uniquely determines the detuning of the augmented triad. A progression like (C, E, A) →(C, E, G♯) →(C, E♭, A♭) can be analyzed as involving a reinterpretation similar to Rameau's *double emploi*, being an elided version of (C, E, A) $\overset{\mathcal{S}}{\longleftrightarrow}$ (C, E, G♯ – ϵ) → (C, E – ϵ, G♯) $\overset{\mathcal{S}}{\longleftrightarrow}$ (C, F, A♭).[22] As a programmer, I regularly use this technique, desymmetrizing chords with a small parameter ϵ that gets set to zero before outputting the results of a musical calculation.

Desymmetrization allows us to define contrapuntal interchanges between transpositionally related chords. Suppose, for instance, we define a contrapuntal interchange $\mathcal{W}$: (C, E, G) ↔ (C, E♭, A♭). This applies to {C, E, G} in two ways, "forward" to A♭, as in (C, E, G) ↔ (C, E♭, A♭), and "backward" to E major, as in (E, G♯, B) ↔ (E, G, C). Here, we need desymmetrization not because the major triad is symmetrical, but because the contrapuntal interchange is inherently ambiguous. We can disambiguate it by writing $\mathcal{W}$: (C, E, G) ↔ (C, E♭ + ϵ, A♭). Any isolated major-triad-in-a-score can be treated either as (C, E, G) or (E,

20 Attributes earlier appeared in connection with Figure 1.1: "C8 considered as C10" assigns the attribute "C10" to the point "C8." The underlying mathematics is the same as in this section.

21 For instance, octave shifts, scalar transposition, and inversion operate on pitch classes while keeping attributes fixed; permutation maps one (attribute, element) pair to another.

22 Desymmetrization also allows us to represent chords with doublings, transforming multisets like {0, 0, 4, 7} into ordinary sets like {0, 0 + ϵ, 4, 7}; using attributes, we could write {(root_1, C), (root_2, C), (third, E), (fifth, G)}. This allows us to restrict our discussion to "bijective voice leadings" in which each pitch class in one chord is mapped to a unique pitch class in another. From this point of view, the set {0, 0 + ϵ, 4, 7} is unrelated to {0, 4, 4 + ϵ, 7}. Modeling this relationship is a topic for another time.

Figure 5.1. Some examples of the contrapuntal interchange $\mathcal{W}$: (C, E, G) ↔ (C, E♭ + ϵ, A♭), combined with transposition down by semitone. Bob Dylan's progression replaces the dominant triad with its relative transform.

G + ϵ, C). However, any given instance of $\mathcal{W}$ is nonambiguous: Desymmetrization ensures that (C, E, G) → (C, E♭, A♭) can only be an instance of $\mathcal{W}$ if it starts with (C, E, G) whereas (C, E, G) → (B, E, G♯) can only be an instance of $\mathcal{W}$ if it starts with (C, E, G + ϵ). This in turn guarantees that (C, E♭, A♭) → (C, E, G) → (B, E, G♯) *cannot* be analyzed as two successive instances of $\mathcal{W}$, as that would require the middle chord to be both (C, E, G) and (C, E, G + ϵ). The desymmetrized $\mathcal{W}$ necessarily alternates between ascending and descending major-third root progressions. Figure 5.1 shows this schema in Beethoven's *Waldstein*, Goffin and King's "Natural Woman," Led Zeppelin's "Kashmir," and the Eagles' "Hotel California"; it also includes a variant from Bob Dylan that replaces dominants with mediants.[23]

I hear this "*Waldstein* sequence" as articulating I–V units over a descending chromatic line, with the dominant chords' harmonic tendency overridden by the chromatic voice leading. This deceptive character is more obvious when the sequence is translated into diatonic space. Define the diatonic contrapuntal interchange $\mathcal{D}$: (C, E, G) ↔ (C, E + ϵ, A); transposed along the diatonic scale it becomes (D, F, A) ↔ (D, F + ϵ, B) and (E, G, B) ↔ (E, G + ϵ, C). Figure 5.2 adds transposition to produce the descending-third "Pachelbel" sequence,

23 In the *Nashville Skyline* recording, the chromatic line is played in the upper register by the organ and pedal steel.

Figure 5.2. The diatonic contrapuntal interchange 𝓓: (C, E, G) ↔ (C, E + ϵ, A) produces the Pachelbel sequence.

sometimes called the Romanesca.[24] The analysis identifies this schema as an instance of a much more general technique appearing in many different styles: Like Figures 0.2a–d it combines parallel voices with an additional "mobile" voice moving between two vertical configurations.[25] This in turn helps *explain* the schema rather than simply treating it as an idiom (Tymoczko 2023c: chap. 1). We could also use 𝓓 to model an equivalence class, the "triadic figured-bass consonance," which has two fixed elements (the bass and the third above it) and one mobile element (either a more-stable fifth or a less-stable sixth).[26]

Inverting 𝓦 around C produces 𝓦′: (C, A♭, F) ↔ (C, A + ϵ, E), which we can rewrite as 𝓦′: (A, C, E,) ↔ (A♭, C + ϵ, F), changing the chords' order and rearranging their notes. Since 𝓦 and 𝓦′ are related by inversion, they are instances of the same inversional contrapuntal interchange 𝓦, and hence a single transformation.[27] Figure 5.3 shows how this transformation appears in the second half of *The Rite of Spring*. The analysis proposes that the minor triad {C, E♭, G} is sometimes reconceived as {C, E♭, G + ϵ} and vice versa; these reinterpretations change the position of the primary melodic voice (open noteheads). Contrapuntal interchanges thus describe a chromatic practice found in Beethoven, Stravinsky, and rock, whose diatonic version appears in almost all triadic music. In each case we have stepwise motion harmonized by alternating vertical configurations.

Satie's hierophantic *Vexations* (1893) distributes eighteen chords over thirteen beats, divided 4 + 4 + 5.[28] The main phrase A appears in two variants: one sounding only the bass (B), and the other transposing the top voice down by octave (C). The performer is instructed to repeat the arrangement BABC eight hundred and forty times over the course of more than twelve hours. The top voice saturates chromatic space between D5 and A5, generating the lower voices from a trio of contrapuntal interchanges (Figure 5.4). The upper voices are largely in

24 Hook 2002 reports that Clough 2000 formalized the Pachelbel progression using UTTs, somewhat along the lines I have done here. Hook 2023 (140) analyzes diatonic sequences using a harmonic *Tonnetz* that, unlike contrapuntal interchanges, does not imply any particular voice leading.

25 See Tymoczko 2011b and 2023c. Other discussions of sequences include Bass 1996; Moreno 1996; Harrison 2003; Ricci 2004; Sprick 2018; Waltham-Smith 2018; and Frederick 2019.

26 Zarlino (1558) 1968: 188, writes, "Musicians often write the sixth *in place of* the fifth, and this is fine" (my italics).

27 I consider this an interesting theoretical point rather than a perceptually salient one.

28 Thanks to Jason Yust for pointing me to this piece.

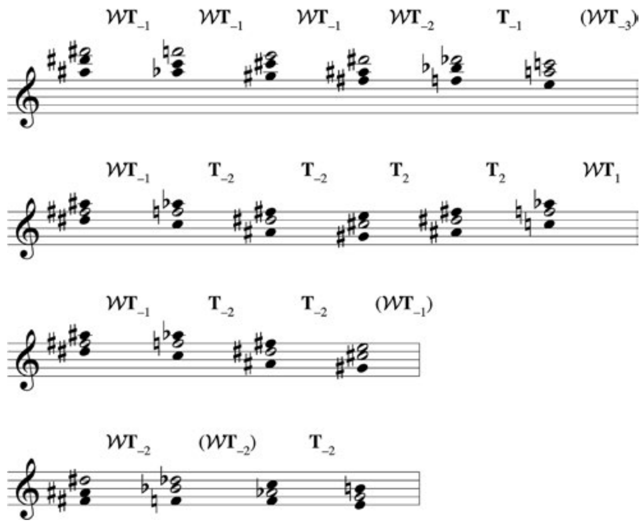

Figure 5.3. The contrapuntal interchange $\mathcal{W}$**: (C, E♭, G) ↔ (B, D♯ + ϵ, G♯) as it appears at R80 of** ***The Rite of Spring*****. Transformations in parentheses mark moments of reinterpretation where a transposition of G♯, B, D♯ + is reinterpreted as G♯, B, D♯ or vice versa. This changes the location of the primary melodic voice (open noteheads).**

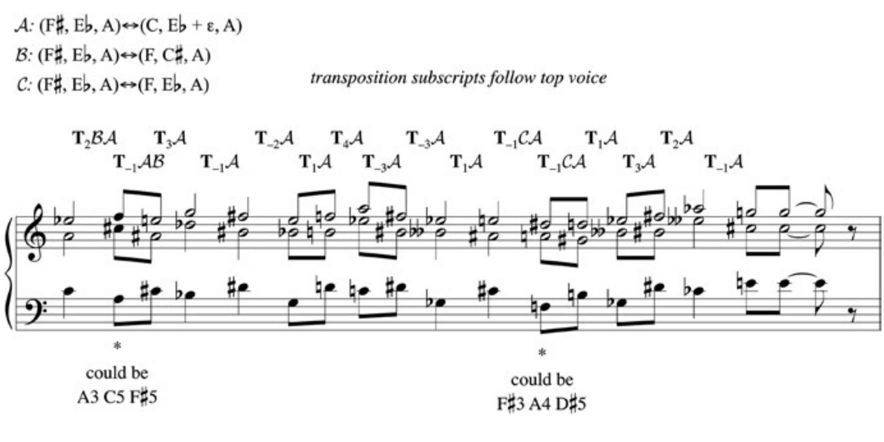

Figure 5.4. Satie's ***Vexations*****. Open noteheads show parallel tritones.**

parallel tritones, with the bass alternately a minor tenth and a diminished fourteenth below the soprano; this can be modeled by the contrapuntal interchange $\mathcal{A}$: (F♯, E♭, A) ↔ (C, E♭ + ϵ, A). (Satie describes the bass as the "theme," but I suspect this is misdirection; the top voice is more conventionally melodic, and it seems a more likely starting point for the composition.) The core oscillation is distorted by two additional contrapuntal interchanges: The first, $\mathcal{B}$: (F♯, E♭, A)

$\leftrightarrow$ (F, C♯, A), relates diminished and augmented triads by stepwise displacement of the lower voices; the second, $\mathcal{C}$: (F♯, E♭, A) $\leftrightarrow$ (F, E♭, A), relates 036 and 026 trichords by single-semitone displacement of the bass. These three contrapuntal interchanges suffice to generate the lower voices from the melody; in this sense they "compress" the musical "data" by providing a compact description of its structure. (Note that because I have defined the three contrapuntal interchanges so as to preserve the top voice, the transpositional subscripts always follow the melody's intervals.) In future work I hope to pursue this connection between musical analysis, encoding, and data compression.

All of these analyses can be considered broadly neo-Riemannian in character, using contrapuntal interchanges to describe oscillations between two different vertical configurations. Yet none of them feature the classic neo-Riemannian case of an involution connecting major and minor triads; instead, they involve either unrelated harmonies (e.g., Figures 4.2–3), or two transpositions of the same chord type. By replacing contextual inversions with contrapuntal interchanges, we open the door to a wider range of progressions, deemphasizing both inversion and set-class.

6. Contrapuntal Interchanges and Equivalence Relations

As intervals, voice-leading transformations are limited in scope: There is, for example, no way to apply a transpositional voice-leading schema $A \rightarrow B$ to a chord that is not transpositionally related to A. This is analogous to the fact that the operation "permute agogo and bongos" cannot be applied when percussionists are playing drum, egg shaker, and flexatone (Section 2). Intervals are *local* objects rather than global functions; in this respect they are ontologically different from symmetries.

One important function of contrapuntal interchanges is to create musical *equivalence classes* allowing us to transport voice-leading transformations between different harmonies. These equivalence classes in turn allow us to model intuitions of similarity—for instance, that major, minor, augmented, and diminished triads are structurally similar objects with roots, thirds, and fifths. The contrapuntal interchange $\mathcal{P}$: (C, E, G) $\leftrightarrow$ (C, E♭, G) tells us which notes in the two triads are analogous, information that allows us to move voice leadings between them.[29] For example, the transpositional voice-leading schema Y: (C, E, G) $\rightarrow$ (B, F, G) maps the major triad to its incomplete dominant seventh by stepwise voice leading. We can use the parallel transformation to define a new transpositional voice-leading schema Y$\mathcal{P}$: (C, E♭, G) $\rightarrow$ (B, F, G), which maps a minor triad to *its* incomplete dominant by stepwise voice leading. Intuitively, Y$\mathcal{P}$ is "the version of Y that operates on minor triads." The contrapuntal interchange $\mathcal{P}$ allows us to formalize this intuition: For any transpositional voice-leading schema S involving a major or minor chord, S$\mathcal{P}$ defines an analogous schema operating on the parallel triad. This is a *nondualist* model of major and

29 In principle we can use a pair of retrograde-related transpositional voice-leading schemas for this purpose as well.

minor: $\mathrm{Y}\mathcal{P}$ is a *perturbation* rather than inversion of Y, lowering the third while keeping everything else the same.

In this context, it is important to avoid multiple contrapuntal interchanges connecting the same two set-classes. First, because they are unnecessary: Given a contrapuntal interchange $A \leftrightarrow B$, and knowing how a voice-leading transformation applies to chord A, we can apply that transformation to all chords in the set-class B. Second, because they create ambiguity. In the previous paragraph, we saw how to apply Y: (C, E, G) $\rightarrow$ (B, F, G) to the minor triad, using an equivalence class generated by the contrapuntal interchange $\mathcal{P}$. But suppose we add the contrapuntal interchange $\mathcal{R}$: (C, E, G) $\leftrightarrow$ (C, E, A) to our equivalence class; now the transformation Y-as-applied-to-minor is both $\mathrm{Y}\mathcal{R}$ or (C, E, A) $\rightarrow$ (B, F, G), and $\mathrm{Y}\mathcal{P}$ or (C, E, A) $\rightarrow$ (D, E, G♯). The two contrapuntal interchanges provide different interpretations of Y-as-applied to minor; what should be univocal is now ambiguous. For this reason we typically want to represent musical equivalence classes using *acyclic* graphs (or *trees*) in which vertices are set-classes, edges are contrapuntal interchanges, and each object represents a unique set-class.[30] Indeed, any connected acyclic graph whose edges are contrapuntal interchanges, and whose objects contain no more than one instance of any set-class, will define a mathematically viable equivalence class.

It is instructive to extend this major-and-minor-triad equivalence class to include more chords and transformations. We start with a set of transpositional voice-leading schemas applying only to the major triad: $\mathbf{T}_x$ is chromatic transposition by x semitones, t_x is transposition along the major triad by x intrinsic steps, and c_0 is the voice exchange (C, E, G) $\rightarrow$ (E, C, G), moving root up by four semitones and third down by four semitones. The remaining pairwise voice exchanges c_x are defined as $\mathrm{t}_{-x}\mathrm{c}_0\mathrm{t}_x$: c_1 moves root down by five semitones and fifth up by five semitones, while c_2 moves third up by three semitones and fifth down by three semitones (Figure 6.1). The voice leadings c_x generate all the voice exchanges connecting a major triad to itself; $\mathbf{T}_x$ and t_y generate all the spacing-preserving voice leadings between major triads. This means that $\mathbf{T}_x$, t_y, and c_z generate every three-voice voice leading among complete major triads.[31]

We can extend this alphabet to the other three triads using the contrapuntal interchanges $\mathcal{P}$: (C, E, G) $\leftrightarrow$ (C, E♭, G), $\mathcal{P}_1$: (C, E, G) $\leftrightarrow$ (C, E♭, G♭), and $\mathcal{P}_2$: (C, E, G) $\leftrightarrow$ (C, E, G♯ – ϵ), detuning the augmented triad as discussed in Section 5. $\mathcal{P}_1$ and $\mathcal{P}_2$ are analogues to the parallel transform, holding root fixed while mapping third to third, and fifth to fifth. These combine to form a complete set of contrapuntal interchanges connecting all four triadic set-classes (Figure 6.2). For instance, the transform $\mathcal{P}_1\mathcal{P}$ takes us from minor to diminished, while $\mathcal{P}\mathcal{P}_1$ takes us in the opposite direction; these are two halves of a contrapuntal interchange which we can call $\mathcal{P}_1\mathcal{P}/\mathcal{P}\mathcal{P}_1$. We can use these transformations to apply $\mathbf{T}_x$, t_y, and

30 Graph cycles can define nontrivial transformations from a chord to itself, and hence create problematic ambiguity.

31 To explore non-bijective voice leadings with more voices we would need to add additional letters to our transformational alphabet; this is a subject for another article.

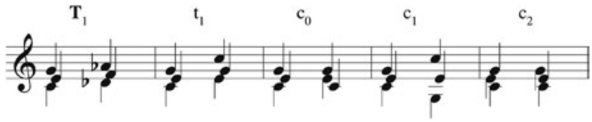

Figure 6.1. Five voice-leading transformations: chromatic transposition ($\mathbf{T}_x$), transposition along the chord (t_y), and three pairwise voice exchanges (c_0, c_1, and c_2).

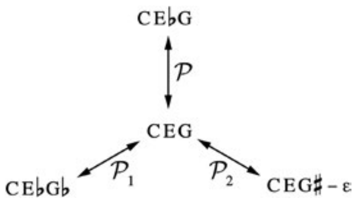

Figure 6.2. A network of contrapuntal interchanges generated by $\mathcal{P}$, $\mathcal{P}_1$, and $\mathcal{P}_2$. These combine to link all four triads to one another.

c_z to every triad: We define $\mathbf{T}_x$, as applied to the minor triad as $\mathcal{P}\mathbf{T}_x\mathcal{P}$and similarly for the other chords and transformations. (Note that this approach makes no use of the inversional relationship between major and minor, nor of the fact that they are related by diatonic transposition.) Readers can check that everything works out: $\mathbf{T}_x$ transposes chromatically by x semitones, t_y transposes along the chord by y intrinsic steps, and c_z crosses analogous elements of each chord—that is, c_0 crosses root and third, c_2 crosses third and fifth, and c_1 crosses root and fifth.[32]

Collectively, the letters $\mathbf{T}_x$, t_y, c_z, $\mathcal{P}$, $\mathcal{P}_1$, and $\mathcal{P}_2$ form a complete voice-leading alphabet generating *all* the three-voice voice leadings among complete triads. This terminology is independent of exact intervallic structure: For instance, the transformation t_1c_1 (or t_1c_{rf}) transforms a root-position close-position triad into a second-inversion open-position triad regardless of whether we start with major, minor, diminished, or augmented. A mathematician would say that the diagram in Figure 6.3 "commutes": We get the same result whether we apply our contrapuntal interchanges before or after our voice leading transformations.[33] This allows us to treat triads as generic objects whose precise constitution remains

32 For this reason, it might be more intuitive to call these transformations c_{rt}, c_{tf} , and c_{fr}, though this notation would not generalize to other chord types.

33 The voice exchanges c_x are defined relative to a specific "reference" ordering of a chord: c_0 is the pairwise exchange of the first two notes in the reference ordering while $c_x = t_{-x}c_0t_x$. The contrapuntal interchanges $\mathcal{P}_i$ thus commute with the voice crossings trivially, by defining what it means to apply "the same" voice crossing to a set-class.

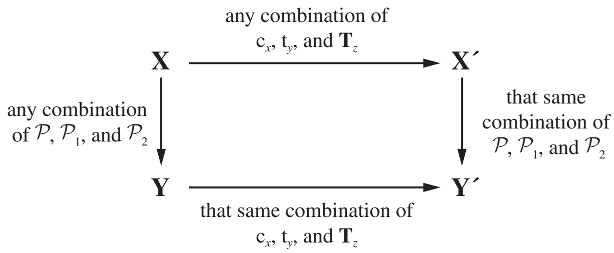

Figure 6.3. A commuting diagram of contrapuntal interchanges and voice-leading transformations.

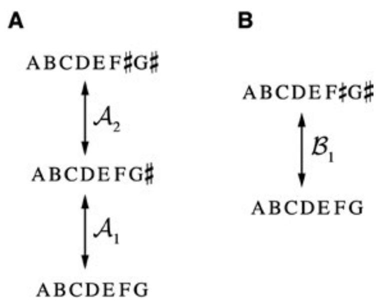

Figure 6.4. Contrapuntal interchanges can be used to model scales with variable degrees such as the Western minor (A) or the Turkish Bûselik makam (B).

unspecified: "a root-position triad in the open voicing." Our universal alphabet allows us to speak about registral inversions, spacing, chordal elements in each voice, root progressions, and voice leadings, all independent of specific chord forms. The upshot is that we can define the term *triad* even in purely chromatic, nonnotated contexts. Here, "triad" refers to an *approximate chromatic set-class*, not defined in terms of notation or the diatonic scale, but as a chromatic "stack of thirds."[34]

The technique can be used to form equivalence classes corresponding to a wide range of intuitive musical terms. For example, we can use it to model scales with variable degrees, such as the Western minor or the Turkish Bûselik makam (Figure 6.4). Or we can formalize terms such as *quartal trichord*, *quartal tetrachord*, *tertian tetrachord*, and so on. In a recent article, I described an *approximate*

34 See Tymoczko 2023a. Thus, I disagree with Hook's (2023: 9) claim that "whenever we refer to interval sizes or chord types such as 'third,' 'ninth,' 'triad,' or 'seventh chord,' we are implicitly gauging pitch relationships in a generic space." To be fair, I would have agreed with Hook a few years ago.

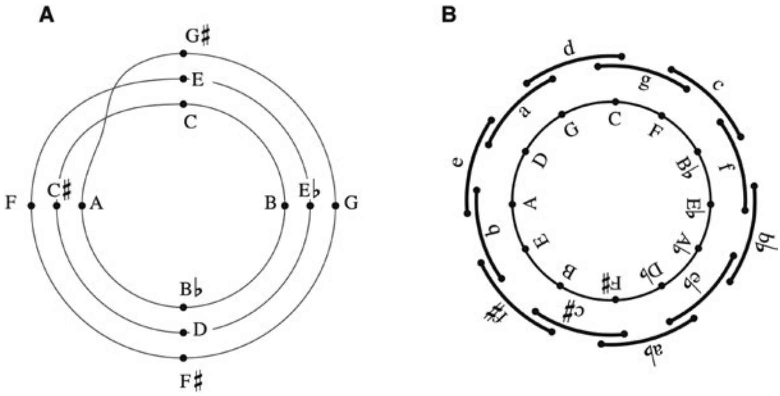

Figure 6.5. A. A voice-leading graph in which points represent major-and-minor-triad pairs related by the $\mathcal{P}$ transform. B. A voice-leading graph representing major and minor keys. Minor keys are spatially extended, stretching between their parallel and relative majors.

set theory based on these categories (Tymoczko 2023a); though I did not realize it, contrapuntal interchanges are a natural tool for formalizing them.[35] The same mechanism could be used in atonal theory to create equivalence classes of multiplicatively or Z-related sets.[36] Category construction is a fundamental music-theoretical activity, important in virtually every genre, style, and tradition.

These categories can be represented by geometrical models in which points represent multiple chords. Figure 6.5 shows two diagrams from *Tonality: An Owner's Manual* (Tymoczko 2023c). In the first, points represent major or minor triads related by the $\mathcal{P}$ transform; parallel triads are close enough to be represented by the same point.[37] The second diagram shows the voice-leading space for major and minor keys, modeled as scales in which one note is singled out as tonic; minor keys are modeled using the trio of scales in Figure 6.4a. Natural and ascending melodic minor are far enough apart to create spatially extended regions in voice-leading space, so that a single minor key can be simultaneously close to its relative and parallel major keys (e.g., the key of A minor stretches from A major to C major). Spatial extension is a novel feature of the geometry of

35 Approximate sets allow us to understand important facts about voicings—for instance, that quartal pentachords are spaced most evenly when in open position, whereas tertian pentachords are spaced as thirds when in the "drop 2" voicing (Tymoczko 2023a).

36 A particularly interesting equivalence class groups together all set-classes residing on a line segment from the perfectly even chord to the boundary of set-class space, defining a form of "multiplicative equivalence." This construction can be used to form "cross sections" of set-class space containing chords with a given smallest interval (Tymoczko 2020b). This means we can discard one dimension of the space.

37 Moving along the spiral represents chromatic transposition while a "loop" around the center of the space, which necessarily leaves the spiral at some point, represents transposition along the chord, clockwise for descending and counterclockwise for ascending (Tymoczko 2023c). The voice exchanges are not represented.

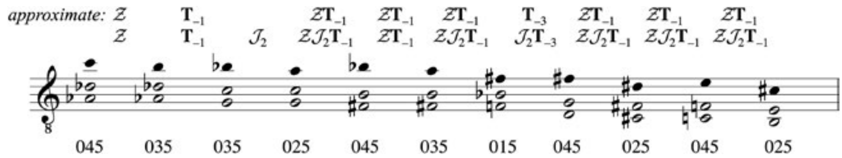

Figure 6.6. Jon Russell's (2018) analysis of the opening sonorities of *The Rite of Spring*, formalized using the contrapuntal interchanges $\mathcal{Z}$: (C, E♭, F) ↔ (C, E, F) and the transformation $\mathcal{J}_2$: (C, E♭, F) ↔ (C, D, F) or (C, E, F) ↔ (C, D♭, F). The approximate analysis simply ignores the $\mathcal{J}_2$ transform, treating the 045 and 035 sets as equivalent ("third-step" trichords).

equivalence classes; here it reconciles circle-of-fifths logic with the importance of the parallel and relative modulations.

A conception of "normal form" allows us to form equivalence classes containing *all sets of a given cardinality*, trees whose objects are set-classes and whose arrows are normal-form-preserving contrapuntal interchanges. As John Roeder (1984) observed, these equivalence classes can be used to define a universal voice-leading alphabet, combining transposition along the scale $\mathbf{T}_x$, transposition along the chord t_y, voice exchanges c_x, and normal-form-preserving contrapuntal interchanges linking different set-classes.[38] The first three of these are intervallic transformations, while the last is an ad hoc equivalence class allowing us to transport voice leadings from point to point (a "wide-tree subgroupoid" in mathematical parlance). In 2020, I showed that the chord-preserving letters of Roeder's alphabet correspond to geometrical features of set-class space: The voice crossings c_z are produced by interactions with mirrors, while the transpositions $\mathbf{T}_x$ and t_y involve the circular dimension (Tymoczko 2020b). A voice leading, or path through set-class space, thus spells out a word in Roeder's alphabet. We can now understand why normal-form-changing voice leadings do not correspond to features of chord space: Normal form is rooted in convention rather symmetries—a useful convention, but different in kind from the other transformations. It makes sense that this convention would be represented by contrapuntal interchanges, an ad hoc transformation, rather than symmetries or intervals.

To illustrate how one might use these ideas analytically, consider two inversional contrapuntal interchanges: $\mathcal{Z}$: (C, E♭, F) ↔(C, E, F) creates an equivalence between two "equipollent" or "third-step" trichords, connecting them by single-semitone voice leading; $\mathcal{J}_2$: (C, E♭, F) ↔(C, D, F) is a generalized neo-Riemannian voice leading connecting inversionally related 025 and 035 trichords. The equivalence relation $\mathcal{Z}$ allows us to define $\mathcal{J}_2$-as-applied-to-015 as $\mathcal{Z}\mathcal{J}_2\mathcal{Z}$, or (C, E, F) ↔ (C, D♭, F), which I will also write as $\mathcal{J}_2$. Figure 6.6 uses these transformations to formalize Jon Russell's 2018 analysis of the opening of *The Rite of Spring*.

38 Roeder used traditional pitch-class intervals rather than paths in pitch-class space. Nevertheless, his prescient work anticipated voice-leading geometry in many ways. The different approaches to normal forming are not important here (Schuijer 2008; Callender, Quinn, and Tymoczko 2008).

Intuitively, the passage features a series of "third-step" trichords, with the top voice mobile and the bottom voices always forming a perfect fourth. Alternatively, one could ignore the *Z* transformations to work with approximate sets: From this perspective, the passage oscillates between "upper voice almost an octave above the top of the fourth" and "upper voice a bit more than an octave above the bottom of the fourth"—a more intuitive and perhaps realistic description of the passage. Contrapuntal interchanges feature prominently in Stravinsky's early music, being closely related (or perhaps even equivalent) to what Russell calls "kaleidoscopic oscillation" (see Figures 0.2d–e, 4.3, 5.3).

A longstanding music-theoretical question concerns the origin of harmonic species: whether they are grounded purely in symmetries or whether they are partly conventional.[39] This is analogous to the question, discussed earlier, of whether *transformations* are necessarily associated with symmetries, or whether they can be ad hoc operations that are purely conventional (Section 2, Section 4). Philosophically, this is a choice between mathematics and practice as the ground of music-theoretical concepts. Theorists have often sided with mathematics, celebrating symmetry as the central music-theoretical principle.

That approach eventually left twentieth-century musicians with an uncomfortable multitude of chromatic set-classes: 29 distinct and unrelated tetrachords, 38 distinct and unrelated pentachords, and so on. Symmetry-oriented theorists tried to restore order by proposing new and increasingly difficult-to-perceive symmetries such complementation, the Z-relation, chord-multiplication, and many others (Morris 1982; Mazzola 2002). (Other theorists proposed various ad hoc systems based on phenomena like complementation and set inclusion, with only a tenuous relation to musical practice [e.g., Forte 1973, 1988].) This search for symmetries contributed to a divergence between American and European traditions, with the former emphasizing set-theory's symmetry-based categories, and the latter emphasizing more intuitive and conventional methods of musical organization (Schuijer 2008: 1–3). Here again, modernist ideology sought to ground music theory in objective mathematics rather than human behavior. Contrapuntal interchanges shift the balance in the other direction, providing rigorous definitions of terms such as "clustered," "tertian," and "quartal." They also allow us to define piece-specific equivalences such as "a perfect fourth with one note inside it" (Figure 6.6) or "a major triad plus one note that is a tritone away from a chord tone" (e.g., Figure 0.2d, which combines 0147 and 047A tetrachords). What results is a more intuitive and humane perspective on the chromatic universe.

39 This debate stretches back to the conflict between figured-bass and root-functional interpretations of consonances, the latter being based on the symmetries of octave displacement and diatonic transposition (Lester 1974; Christensen 1993); also relevant is Riemann's attempt to reconceive Zarlino's theory as based on the symmetry of inversion (Rehding 2003: 28–29). The debate continues to the present day, with contemporary writers like Robert Gjerdingen (2007) and Ian Quinn (2018) deemphasizing symmetry by problematizing the harmonic/nonharmonic distinction and advocating for figured-bass categories.

7. Dual Transpositional Systems

We have now described voice-leading transformations and contrapuntal interchanges, identifying some contexts in which they can be useful. The rest of the article will consider the problems arising from their combinatorial mechanics. We start with the system that results from the unrestricted combination of contrapuntal interchanges and ordinary transposition; the next section considers approaches that reject either ordinary transposition or contrapuntal interchanges. We end by exploring the rejection of transformational closure.

A *dual transposition system* is a collection of transformations that can be represented as ordered pairs of transpositions combining component-wise: $(\mathbf{T}_z, \mathbf{T}_w)(\mathbf{T}_x, \mathbf{T}_y)=(\mathbf{T}_{x+z}, \mathbf{T}_{y+w})$ (O'Donnell 1997). A dual transposition $(\mathbf{T}_a, \mathbf{T}_a)$ acts like an *ordinary transposition* and hence can be written $\boldsymbol{T}_a$; analogously, a dual transposition $(\mathbf{T}_b, \mathbf{T}_{-b})$ acts like a Schritt and can be written $\boldsymbol{S}_b$. Any dual transposition $(\mathbf{T}_x, \mathbf{T}_y)$ can be written as the combination of an ordinary transposition and a Schritt, $\boldsymbol{T}_a\boldsymbol{S}_b$: setting $x=a+b$ and $y=a-b$, we obtain $a=(x+y)/2$ and $b=(x-y)/2$.[40] Thus $(\mathbf{T}_x, \mathbf{T}_y)=\boldsymbol{T}_{(x+y)/2}\boldsymbol{S}_{(x-y)/2}$ and $\boldsymbol{T}_a\boldsymbol{S}_b=(\mathbf{T}_{a+b}, \mathbf{T}_{a-b})$.

This decomposition is useful both inside and outside of music theory. In audio engineering, it decomposes the motion of two speakers $(\mathbf{T}_L, \mathbf{T}_R)$ into in-phase $(\boldsymbol{T}_a)$ and anti-phase components $(\boldsymbol{S}_b)$, facilitating the separation of sound sources, the removal of inaudible DC components, and the artificial widening of the stereo field. In physics, it decomposes the motion of two particles into the motion of their shared coordinate system $(\boldsymbol{T}_a)$ and their relative motion within that system $(\boldsymbol{S}_b)$. In twelve-tone theory it transforms one inversionally combinatorial hexachord into another by moving its whole-tone components independently (Figure 7.1; see Morris 1982 and Mead 1991). In atonal set theory, it redescribes the motion of two superimposed sets using "positive isography" $(\boldsymbol{T}_a)$ and "strong isography" $(\boldsymbol{S}_b)$ (Figure 7.2).[41] As a special case, this allows us to analyze two independent voices as combining parallel $(\boldsymbol{T}_a)$ and contrary $(\boldsymbol{S}_b)$ motion (Tymoczko 2023c: 87–95). In each case, we have two coordinate systems, one using dual transpositions $(\mathbf{T}_x, \mathbf{T}_y)$ and the other combining parallel with contrary motion ($\boldsymbol{T}_a$ and $\boldsymbol{S}_b$). Geometrically, these relate by 45° rotation (Figure 7.3).

Contrapuntal interchanges combine with transposition to generate a dual transpositional system applying different transpositions to different set-classes, transformations $(\mathbf{T}_x, \mathbf{T}_y)$ that transpose set-class A by x semitones and set-class B by y semitones. Julian Hook has called these "uniform triadic transformations," or UTTs (see Hook 2002 and 2023: 317–20 and 324). Suppose we have two transpositional set-classes A and B linked by a contrapuntal interchange $\mathcal{C}$ that determines a default transpositional relation between them. We can define the

40 Note that the division by 2 can produce quarter tones: $(\mathbf{T}_1, \mathbf{T}_0)$ is equivalent to $\boldsymbol{T}_{.5}\boldsymbol{S}_{.5}$, combining quarter-tone transposition with a quarter-tone Schritt.

41 Standard K-net theory does not quantify "strong isography"; Tymoczko 2007 argues it should.

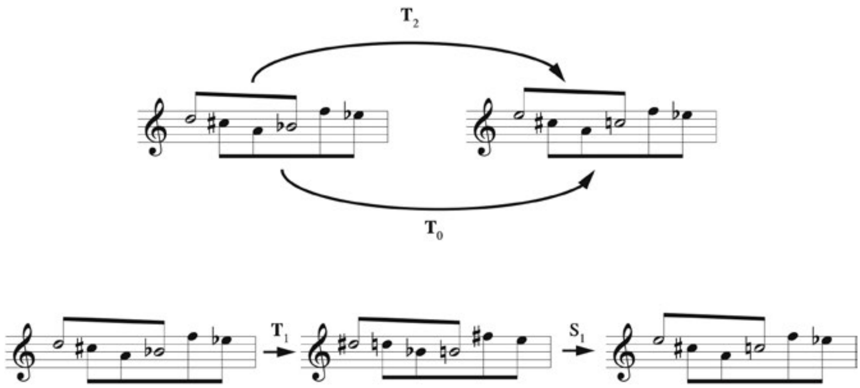

Figure 7.1. **(*top*) Andrew Mead and Robert Morris have described a twelve-tone operation that independently transposes a hexachord's two whole-tone subsets, with both transpositions having the same even/odd parity; these dual transpositions preserve the property of inversional combinatoriality. (*bottom*) We can also represent them as combining chromatic transposition with a Schritt that transposes the two subsets in contrary motion.**

Figure 7.2. **(*left*) Atonal set theorists have used dual transpositions to transform subsets of a composite harmony. (*right*) We can represent these transformations as combining chromatic transposition with a Schritt that moves the two subsets in contrary motion; these two transformations are equivalent to "positive" and "strong" isography.**

UTT $(\mathcal{C}^n, x, y)$ as the transformation that transposes A by x semitones, B by y semitones, and then continues with an n-fold application of $\mathcal{C}$. Thus it acts on A as $\mathbf{T}_x\mathcal{C}^n$ and on B as $\mathbf{T}_y\mathcal{C}^n$. Since $\mathcal{C}^2$, or "interchanging twice," does nothing, this means we apply $\mathcal{C}$ once $(\mathbf{T}_x\mathcal{C}/\mathbf{T}_y\mathcal{C})$ or not at all $(\mathbf{T}_x/\mathbf{T}_y)$. In Hook's work, A and B are typically major and minor triads and $\mathcal{C}$ the parallel transform, understood harmonically; for us, A and B can be any two transpositional set-classes and $\mathcal{C}$ any contrapuntal interchange connecting them. We can rewrite any UTT $(\mathcal{C}^n, x, y)$ as $\mathcal{C}^n\mathbf{S}_b\boldsymbol{T}_a$, combining ordinary transposition, a Schritt, and a contrapuntal interchange.

Now recall the point made by Figures 4.4–5: Contrapuntal interchanges $A \leftrightarrow B$ and $A \leftrightarrow \mathbf{T}_x(B)$ compose to produce Schritts that move A by x and B by $-x$. Combining these transformations with transposition, and putting aside specific voice-leading effects, we obtain the UTT subgroup $(\mathcal{C}^n, i + jx, i - jx)$,

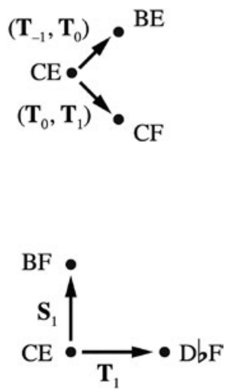

Figure 7.3. The dual-transposition and Schritt/transposition decompositions can be expressed as a 45° rotation of coordinates. In ordinary Cartesian geometry, dual transpositions are typically represented by horizontal and vertical motion; in voice-leading geometry it is often convenient to rotate the axes so that Schritts move horizontally while transpositions move vertically.

for fixed integer x and arbitrary integers i, j, n. By setting i to $c - jx$, we can form a transformation that moves A by any number of semitones c, while moving B by $c - 2jx$ semitones, with $(\mathcal{C}^0, c, c - 2jx)$ preserving set-class and $(\mathcal{C}^1, c, c - 2jx)$ changing it. Since c and $c - 2jx$ are independent, the behavior of A tells us nothing about the behavior of B.[42]

For me, this is problematic: I have a hard time understanding why anyone would want to associate "transposing set-class A by c semitones" with "transposing set-class B by $c - 2jx$ semitones," and I am unaware of any musical style that regularly employs transformations of this sort. Furthermore, I question the description of these transformations as "uniform," since they apply independent transpositions to the two set-classes; this is not "uniformity" in the plain-language sense of "doing the same thing to all the objects in a domain."[43] As a teacher I would be reluctant to ask students to identify such transformations, particularly in an ear-training context. And even if it were possible for us to retrain our ears to hear these transformations, I doubt that this would improve our ability to understand music made by those who had not been through the same reeducation.

42 Note that the difference between the two transpositions is controlled by the value *x*, the transposition relating the destination chord of the two generating contrapuntal interchanges; if *x* generates a proper subgroup of the scale, then not all differences are achievable.

43 I would say that Hook's UTTs act uniformly on major triads, and uniformly on minor triads, but not on triads as such: there is no relation between the major-chord transformation and the minor-chord transformation.

I suspect that many theorists agree with these points even while using UTTs and paying lip service to transformational closure. Transformational closure tells us that if we want both contextual inversions and ordinary transposition, then we need the group they produce (which is, roughly, the UTTs).[44] One can comply with this demand while analyzing music using only that subset of transformations that one finds musically relevant. This is to reject transformational closure in practice but not in principle: It is as if there were two different music theories, an "official" theory that embraced transformational closure and an unofficial analytical practice that rejected it. One of my goals is to bridge the gap between official and unofficial, explicitly confronting the failure of transformational closure. Distinguishing significant from insignificant transformations is among the most important projects a theorist can undertake; it should not be relegated to the realm of untheorized practice.

It should also be said that there is something mathematically strange about combining transpositions and Schritts into a single group: For as we saw in Sections 2 and 3, these are dual manifestations of *one and the same* group, a set of symmetries and their corresponding intervals. We are not required to fuse these dual transformations into a larger group, even if musicians sometimes combine them. It is of course possible to form a larger symmetry group in this way; but this necessarily creates a *new* interval group that is the dual of this new, larger symmetry group; and this new interval group will show up in various practical contexts whether we want it to or not.[45] This is illustrated by Figure 7.4, which shows a tower of groups starting with transposition and inversion, progressing through the UTTs, and then the QTTs (which incorporate absolute inversion), arriving at an even-larger group that has no music-theoretical name. Mathematically, it is impossible to avoid group-theoretical pluralism: We will always confront multiple transformation groups, relating to one another as symmetries do to intervals.

To be clear, I have no objection to combining contrapuntal interchanges with one another, nor with transpositions: I did the former in Figure 6.6 and the latter in Figures 0.2, 0.3, 3.6, and 5.1–4. Nor do I have any principled objection to combining symmetries and intervals; on the contrary, I think this is rather common in music. My point is rather that theorists should be more explicit about the fact that individually comprehensible transformations might sometimes combine to form incomprehensible composites. For while I have strong intuitions about the comprehensibility of particular combinations, I have no principled account of those feelings. Why is it that some combinations of transformations seem musically sensible while others do not? Our lack of answers reflects our reluctance to probe the foundations of the transformational paradigm.

44 For reasons discussed in n. 40, twelve-tone Schritts combine with ordinary transposition to produce *half* the UTTs.

45 For instance, the dual group will relate analogous nodes in "isographic" networks whose arrows are labeled by UTTs (see Lewin [1987] 2007 on isography).

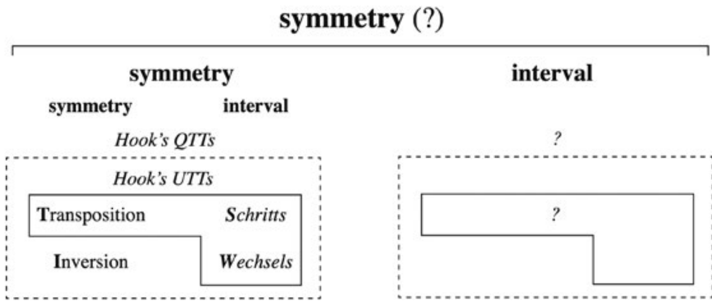

Figure 7.4. **(*left*) Schritts and Wechsels are the intervals associated with the symmetries of transposition and inversion. Hook combines ordinary transposition, Schritts, and Wechsels into a larger group which has its own intervallic dual (the six-sided figure enclosing a question mark on the right). Hook's "quasi-uniform transformations" (QTTs) incorporate inversion, producing an even-larger symmetry group with an even larger intervallic dual. These could be combined into a larger symmetry group (symbolized by "symmetry (?)" at the top of the figure), which would have *yet another* intervallic dual.**

8. Limiting the Transformational Vocabulary

An alternative strategy rejects intuitively useful transformations—for instance, using *either* absolute transpositions and inversions *or* Schritts and Wechsels, but not both. Neo-Riemannian theorists such as Henry Klumpenhouwer (1994, 2000) and Richard Cohn (2012), for example, analyze chord progressions using combinations of *L*, *P*, and *R*, avoiding ordinary transposition in that context. This raises a host of complex problems, the first of which is psychological: I personally find it very hard to conceive of $\mathbf{T}_2$-as-applied-to-minor as being equivalent to $\mathbf{T}_{-2}$-as-applied-to-major. Try as I might, I can hear little substantive resemblance between (C, E♭, G) → (D, F, A) and (C, E, G) → (B♭, D, F), nor am I aware of any pre–twentieth-century musical styles in which these two progressions are syntactic or expressive analogues. Modernist ideology encourages us to stifle such reservations, molding our perceptions to preexisting mathematical templates, at the cost of a substantial gap between music theory and musical practice.

That challenge is exacerbated in the general context where one allows arbitrary contrapuntal interchanges. I can perhaps imagine that musicians might develop a general sensitivity to inversion, helping them to group (C, E♭, G) → (D, F, A) with (C, E, G) → (B♭, D, F). But this capacity would not lead them to link (C, E♭, G) →(C♭, E♭, A♭) and (D♭, E♭, G) → (B, F, G) since minor triads and 026 trichords are unrelated. Yet both (C, E♭, G) → (C♭, E♭, A♭) and (D♭, E♭, G) → (B, F, G) are instances of the combination *XY* (cf. Figure 0.3). This relationship, rather than being underwritten by the general symmetry of inversion, is dependent on the conventional definition of the *X* and *Y* transforms. Any such equivalence would need to be learned piecemeal.

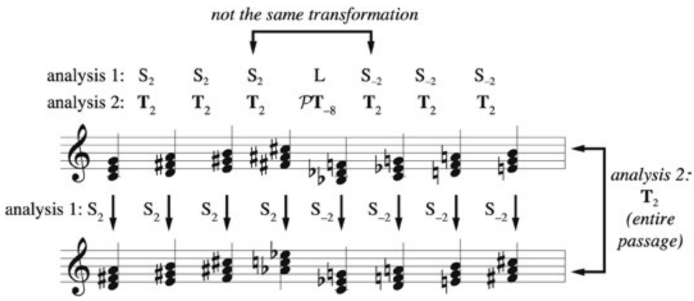

Figure 8.1. Two analyses of the same passage. If we reject ordinary transposition then we need to relate the two measures using multiple Schritts.

A second difficulty is that nondualist transposition is a basic symmetry of human perception: Transposing music leaves it "sounding the same" in some elementary sense. This symmetry is difficult to express using Schritts. From a nondualist perspective the two lines in Figure 8.1 are related by the single transposition $\mathbf{T}_2$, but if we reject transposition then we have to say that the major chords have been acted on by the Schritt S_2 while the minor triads have been acted on by S_{-2}. Even the most committed dualist would acknowledge that this combination of S_x and S_{-x} is special by virtue of acting like the symmetry of ordinary transposition. Such challenges magnify once we start to consider passages with nontriadic material, for Schritts are intervals and hence defined only locally: Knowing how a Schritt affects one set-class does not tell us how it affects another, whereas transpositions are defined for any pitch-class set. Avoiding transposition thus requires us to replace global transposition with local Schritts, and this involves a large number of purely conventional decisions.

Thus it is rare to find theorists completely renouncing ordinary transposition. Cohn, for example, often observes that extended passages are transpositionally related.[46] This creates the uncomfortable situation where different transformations relate the same pairs of chords: In analyzing Figure 8.1, Cohn might use ordinary transposition to relate the lines while using combinations of $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{R}$ to analyze progressions within them. This means that C major $\Rightarrow$ D major pairs would be analyzed differently depending on whether they are horizontal or vertical. I find it counterintuitive to use different transformations in the two circumstances. How does one know when to use a Schritt and when to use an ordinary transposition? And if we can use transposition to relate the two lines of Figure 8.1, then why not use it to analyze progressions like C major $\Rightarrow$ D major?

Alternatively, one could save transformational closure by rejecting contrapuntal interchanges: Rather than a single "$\mathcal{P}$ transform," one could define two

46 See, e.g., Cohn 2012: 27 (both with respect to Figures 2.7 and 2.8), 45, 85, and elsewhere.

separate voice-leading transformations, one sending major to minor ($P_{M\rightarrow m}$ or just P), the other sending minor to major ($P_{m\rightarrow M}$, P inverse, or P^{-1}). By refusing to consider the two directions to be the same transformation, we block the construction of dualist Schritts: Instead of a single transformation acting oppositely on different chords, we would have two *separate* transformations $P_{M\rightarrow m}L_{m\rightarrow M}$ and $P_{m\rightarrow M}L_{M\rightarrow m}$ (or, in mathematicians' notation, PL and $P^{-1}L^{-1}$). There is nothing paradoxical about the fact that different transformations operate differently.

This strategy rejects concretely valuable transformations for abstract combinatorial reasons. I personally find contrapuntal interchanges to be useful when I am sitting at the keyboard, and I have no problem applying them in practical contexts. I like the compact analytic notation they provide (e.g., Figures 0.2, 4.2–3, 5.1–4). The suggestion here is to reject contrapuntal interchanges not because they are inherently problematic but because of how they combine. Once again, this reflects a modernist imperative to prioritize mathematical elegance over musical practice. Unlike the other cases we have considered, however, the mathematics here would serve closed mindedness rather than utopianism: The original goal of transformational theory was to open theorists' minds to the many different processes and relationships that could be embodied in a piece of music. A theory that cannot encompass contrapuntal interchanges hardly deserves the name "transformational."

9. Against Transformational Closure

I have been arguing that the notion of a transformation group is more delicate than many theorists realize. In particular, I have suggested that we regularly encounter three distinct failures of transformational closure. The first is a *practical* failure, in which common and comprehensible transformations, such as $\mathcal{L}$, $\mathcal{P}$, $\mathcal{R}$, and $\mathbf{T}_x$, combine to form composites that are not comprehensible. There is no transformation group that contains both ordinary transposition and contextual inversions, while also being uncluttered by incomprehensible transformations. The second is a *mathematical* failure that results from the fact that a symmetry group generates an isomorphic-but-not-identical group of intervals, with the two groups playing different mathematical roles (Section 2). We should therefore expect group-theoretical pluralism, with the intervals-at-different-points belonging to different groups, and being different from the symmetries. (This pluralism is concretely reflected in the fact that defining a Schritt for one set-class, such as the major triad, does not define it for another, such as the 026 trichord.) The third is a *philosophical* failure resulting from a misattribution of group-theoretical properties to concrete things rather than numbers in the abstract (Section 1). These points unite in my anti-modernist proposal that mathematics is often an unreliable guide to musical significance.

This anti-modernism leads to six more specific claims. First, since musicians use a wide range of transformations, it is not fruitful to limit our analytical vocabulary a priori along the lines of Section 8: We need to remain receptive to

Figure 9.1. Two passages using $\mathcal{H}$, $\mathbf{T}_7$, and $\mathcal{L}$, one starting with major and one starting with minor. One could easily imagine finding both passages in a single piece, and wanting to analyze them as in this example. Doing so need not commit the analyst to the utility or comprehensibility of $\mathcal{H}\mathbf{T}_7\mathcal{L}$.

whatever transformations we might find in actual pieces. Second, the transformations that musicians use, even from one chord to the next, may not combine to produce perceptible or otherwise significant composites. Thus while it may be reasonable to train ourselves to hear individual transformations like $\mathcal{H}$, $\mathbf{T}_7$, and $\mathcal{L}$, it might be unprofitable to try to hear compounds like $\mathcal{H}\mathbf{T}_7\mathcal{L}$ (Figure 9.1). Third, musical spaces have their own logic and structure which can influence our choice of formalism: We might sometimes want to keep symmetries and intervals distinct, even if composers occasionally (or regularly) combine them (Section 2). Fourth, transformations such as $\mathcal{X}\mathbf{T}_y$, which combine a contrapuntal interchange with a transposition, are often easier to understand than combinations of contrapuntal interchanges like $\mathcal{XY}$. Indeed, combinations like $\mathcal{X}\mathbf{T}_y$ form a strand in an extended common practice connecting multiple genres and styles (Figures 0.2, 4.3, 5.1–4, 6.6). Fifth, both set theory and neo-Riemannian theory have privileged *exact intervallic identity* over the flexible relationships that often characterize musical practice—as exemplified by the neo-Riemannian focus on inversion. Contrapuntal interchanges allow for approximate harmonic categories and a wider range of transformational possibilities. Sixth, there is an interesting theoretical project of exploring why some transformational combinations seem to be more comprehensible than others. This question has received relatively little discussion, in part because theorists have reflexively assumed that transformations always form groups.

In *Audacious Euphony*, Richard Cohn (2012: 37) writes that I think "the *system* of [neo-Riemannian] triadic transformations is 'fundamentally dualist' and hence ripe for wholesale rejection." I have italicized the word "system" because I take that to be the crux of the issue. I think $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{R}$ are useful both on their own and in combinations like $\mathcal{R}\mathbf{T}_{-1}$. But Cohn is correct to say that I have problems with the group ("system") formed by their unrestricted combination. This is because that group contains hard-to-perceive Schritts, which are dualist by virtue of transposing major and minor triads in opposite directions.[47] If we are committed to transformational closure, then accepting $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{R}$ forces us to accept $\mathcal{LR}$,

47 Cohn 2012: 37–39 defines "dualism" intensionally, as way of conceiving transformations. For me, however, it is a matter of how transformations behave ("extensionally"): a dualist Schritt transposes inversionally related collections by the same amount in opposite directions; my argument is that these are ipso facto difficult to comprehend, particularly when combined with ordinary transposition. It does not matter how one thinks about the transformations.

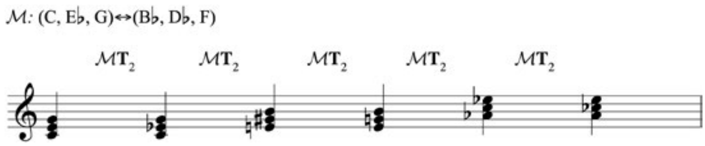

Figure 9.2. It is difficult to understand this a sequence in which each chord is linked to the next by the very same transformation. $\mathcal{M}$ is equivalent to the combination $\mathcal{PRPRL}$.

and accepting ordinary transposition forces us to accept $\mathcal{PRPRL}\,\mathbf{T}_2$ as well (Figure 9.2). I am reluctant to do without transposition or contrapuntal interchanges; and since I have difficulty comprehending transformations like $\mathcal{PRPRL}\,\mathbf{T}_2$, I have no choice but to reject the "system" of neo-Riemannian transformations.

That rejection is tempered by three qualifications. First, I have proposed a framework in which we can use *any contrapuntal interchange whatsoever*, including neo-Riemannian transformations like $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{R}$, "slide," and "hexpole." Second, I have argued that we can use geometrical models like the Tonnetz without embracing transformations like $\mathcal{LR}$: as a space, the Tonnetz is compatible with both the **OPTI** and **OPT** symmetry groups, and if we choose the latter then its paths are transpositional voice-leading schemas; this allows us to treat the Tonnetz nondualistically, as a graph in three-note chord space rather than a map of dualist transformations (Tymoczko 2012). Third, I have argued that Schritts like $\mathcal{LP}$ can be *theoretically* useful, as there are abstract questions whose answers come in inversional pairs (e.g., "what are the semitonal voice leadings between major and minor triads?") (Tymoczko 2011a). Thus I am advocating a *partial* rather than total rejection of neo-Riemannian theory, retaining its transformations while rejecting the group they generate.

This proposal, however, raises an important question. When is it fruitful to combine contrapuntal interchanges with transposition? I have suggested that the analyses in Figures 0.2, 4.2, Section 5 (etc.) are reasonable, but that in Figure 9.2 is less so. All of these combine a single contrapuntal interchange with ordinary transposition. Why is this last example any different from its predecessors?

I believe that the comprehensibility of a transformation like $\mathcal{C}\mathbf{T}_x$ depends on whether $\mathcal{C}$ preserves common tones. When it does, the common tones move in parallel to articulate the transposition $\mathbf{T}_x$, while the remaining voices move relative to those voices, articulating the contrapuntal interchange $\mathcal{C}$. This allows us to understand the combination as a single musical act that moves a flexible vertical genus by x semitones while alternating between its two specific forms.[48] (This is easier when the contrapuntal interchange $\mathcal{C}$ is small, as in Figures 0.2a–d,

48 Again, $\mathcal{C}\mathbf{T}_x$ is a single transformation, just like 3×4 is a single number; the issue is not whether we can disentangle the individual contributions of $\mathcal{C}$ and $\mathbf{T}_x$, but whether we can understand $\mathcal{C}\mathbf{T}_x$ as *a single coherent transformation*.

for then the specific forms can be understood as variants, rather than completely different structures.) When the contrapuntal interchange does not preserve any common tones, it is much harder to interpret the configuration this way: In Figure 9.2, no voice moves consistently, and hence we have no reason to think that a single transformation is being applied.

For this reason, I would argue that common-tone-preserving contrapuntal interchanges are privileged: Transformations like $\mathcal{R}\mathbf{T}_{-1}$ have a musical significance that transformations like $\mathcal{PRPRL}\mathbf{T}_2$ lack. The problem is that the two are *mathematically* similar but *psychologically* different: $\mathcal{LRPRP}$ is itself a contrapuntal interchange that we can label $\mathcal{M}$, and both $\mathcal{R}\mathbf{T}_{-1}$ and $\mathcal{M}\mathbf{T}_2$ combine one contrapuntal interchange with transposition; yet the former combination is considerably easier to understand. Once again, the intellectual virtues of mathematical consistency and psychological realism pull in opposite directions: Mathematics suggests we treat the two transformations the same way, but psychology suggests we treat them differently. Modernism encourages us to favor mathematics, but I prefer to follow psychology.

These ideas might seem radical to twenty-first century theorists, habituated to think of transformations as belonging to groups. But from an embodied perspective they are not so strange. Sequences produced by $\mathcal{R}\mathbf{T}_x$ have a clear contrapuntal structure, with at least one voice repeatedly moving by x semitones; sequences produced by $\mathcal{H}\mathbf{T}_7$ or $\mathcal{PRPRL}\mathbf{T}_2$ do not. One can take that structure to *define* the relevant transformational universe, in which case $\mathcal{H}\mathbf{T}_7$ and $\mathcal{PRPRL}\mathbf{T}_2$ simply do not qualify. It is only when we think abstractly, in terms of algebraic symbols and their combinations, that we start to imagine *all possible* composites of $\mathcal{L}$, $\mathcal{P}$, $\mathcal{R}$, and $\mathbf{T}_x$. This abstract attitude is characteristic of modernism rather than being a cognitive or musical default. Indeed, Hugo Riemann himself avoided combining his transformations: From a paleo-Riemannian perspective, $\mathcal{L}$-of-the-subdominant can act like a subdominant, and $\mathcal{R}$-of-the-subdominant can act like a subdominant, but $\mathcal{LR}$-of-the-subdominant does *not* act like a subdominant, since it is the tonic (Harrison 2011: 570). David Lewin tried to improve this theory by reinterpreting its transformations group-theoretically, but I worry this may have been a step in the wrong direction.[49] Perhaps we should be moving *away* from Riemann's symmetry-focused proto-modernism, rather than pushing it forward.

10. Conclusion

Nothing I have said requires theorists to reject group theory altogether. Symmetries, as invertible operations that leave a space essentially unchanged, naturally

49 Lewin ([1987] 2007: 177) claims that Riemann "did not quite ever realize that he was conceiving 'dominant' . . . as something one *does* to a Klang." But I suspect Riemann may have thought of the dominant more like a place than a transformation: $\mathcal{R}$-of-the-dominant was close enough to that place to sound "dominant-like," while $\mathcal{RL}$ of the dominant was too far away to have dominant qualities. Lewin 1984 (344) proposes that long chains of "subdominant" transforms preserve subdominant quality; if true, this would allow any equal-tempered triad to bear a subdominant relation to any other, so that a chord could potentially be its own subdominant.

form groups: The composition of two structure-preserving transformations is itself structure preserving. And symmetries give rise to a dual set of transformations that also form groups of loops at a point. But the importance of groups, in these specific contexts, does not justify a general presumption that transformations always form groups, much less the even-stronger principle of transformational closure. For music also involves a range of ad hoc transformations that need not have the same mathematical properties. These are individually significant musical moves, meaningful in some specific context, whose combinations are not necessarily meaningful. Contrapuntal interchanges are interesting, in part, because they provide such a clear example of transformations that are analytically useful and yet group-theoretically problematic.

In other words, groups and symmetries are useful in some but not all circumstances. We can add or subtract temperatures so long as we are dealing with small changes far from absolute zero. We can talk about "the group of transpositions" so long as we are far from the bounds of pitch perception. Ultimately, there is no such thing as an infinitely high pitch or an infinitely low temperature. But idealization is a powerful intellectual tool, and almost all theories use it to some degree. The question is not, Do transformations *always* form groups? or Do transformations *ever* form groups?, but rather, In what circumstances can groups be useful? That is a specific example of a more general question that I think theorists should be asking, namely, What is the relation between mathematical abstraction and embodied musical practice?

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